A Note on Strictly Competitive Games

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Abstract. Strictly competitive games are a class of 2-player games often quoted in the literature to be a proper generalization of zero-sum games. Other times it is claimed, e.g. by Aumann, that strictly competitive games are only payoff transformations of zero-sum games. But to the best of our knowledge there is no proof of such claim. We shed light to this point of confusion in the literature, showing that any strictly competitive game is indeed a payoff transformation of a zero sum-game; in fact, an affine transformation. We offer two proofs of this fact, one combinatorial and one algebraic.

1 Introduction

A two-person game is *strictly competitive* [1] if it has the following property: if both players change their mixed strategies, then either there is no change in the expected payoffs, or one of the two expected payoffs increases and the other decreases. That is, all pairs of mixed strategies are Pareto optimal. Mathematically, a game $(A, -B)$ is strictly competitive if for any two pairs of mixed strategies (x, y) and (x', y'), $x^{\mathrm{T}} A y - x'^{\mathrm{T}} A y'$ and $x^{\mathrm{T}} B y - x'^{\mathrm{T}} B y'$ have the same sign. [4] Obviously, these games generalize zero-sum games (the case $A = B$). The question is, how much more general than zero-sum games is this class?

There is much confusion in the literature about this question. Aumann writes *"Recall that a strictly competitive game is defined as a two-person game in which if one outcome is preferred to another by one player, the preference is reversed for the other. Since randomized strategies are admitted, this condition applies also to mixed outcomes (probability mixtures of pure outcomes). From this it may be seen that a two-person game is strictly competitive if and only if, for an appropriate choice of utility functions, the utility payoffs of the players sum to zero in each square of the matrix."*

Notice that "appropriate choice" is not defined, and no proof, or outline, is given. Aumann's insight above is mirrored elsewhere in the literature, e.g. in the textbooks, [3, 4], also without proof. Elsewhere, in lieu of proof a rather straightforward weaker fact is pointed out: Let $a_1, \ldots, a_n$ be n events and let $u(a_i), v(a_i)$ be the utilities of players 1 and 2 respectively. Now suppose that for any pair of probability vectors p, q we have $\sum p_i u(a_i) \geq \sum q_i u(a_i)$ iff $\sum p_i v(a_i) \leq$

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[4] For our purposes the *sign* function takes on three values, $+$, $-$, and 0.

$\sum q_i v(a_i)$. Then it is easy to see that there exist affine transformations of u, v, call them u', v', such that $v'(a_i) = -u'(a_i)$ for all i. Aumann's assertion is stronger: Its hypothesis is that the inequalities hold for all distributions *that are products of mixed strategies.* To increase the confusion, in [5], strictly competitive games are defined with x, y, x', y' above restricted to pure strategies—this is a proper generalization, albeit of no interest. And elsewhere, strictly competitive games are treated as a proper generalization of zero-sum games.

In this note we prove Aumann's assertion. In fact, we give two very different proofs, one combinatorial and one algebraic (both are quite nontrivial). Let A and B be $m \times n$ real matrices. By Δ_n we denote all distributions (mixed strategies) over $[n]$. We say that matrix B is an *affine variant* of matrix A if for some $\lambda > 0$ and unrestricted μ, $B = \lambda \cdot A + \mu \cdot U$, where U is $m \times n$ all-ones matrix. Our main result is the following:

Theorem 1. *If for all $x, x' \in \Delta_m$ and $y, y' \in \Delta_n$, $x^{\mathrm{T}} A y - x'^{\mathrm{T}} A y'$ and $x^{\mathrm{T}} B y - x'^{\mathrm{T}} B y'$ have the same sign, then B is an affine variant of A.*

Note that the converse is trivial.

2 A Combinatorial Proof

Consider the strictly competitive game $(A, -B)$ with at least two pure strategies for each player. Let

$$a_{max} = \max_{ij} A_{ij} \ , \ a_{min} = \min_{ij} A_{ij}$$

$$\text{and} \quad b_{max} = \max_{ij} B_{ij} \ , \ b_{min} = \min_{ij} B_{ij}.$$

Lemma 1. *For all i, j:*

$$a_{ij} = a_{max} \quad \Longleftrightarrow \quad b_{ij} = b_{max}; \tag{1}$$

$$a_{ij} = a_{min} \quad \Longleftrightarrow \quad b_{ij} = b_{min}. \tag{2}$$

Proof. We only show the first assertion. The other assertion can be shown similarly. Suppose there exist i, j such that $a_{ij} = a_{max}$, but $b_{ij} < b_{max}$. Let then k, ℓ be such that $b_{k\ell} = b_{max}$. If x, x' are the pure strategies i, k and y, y' the pure strategies j, ℓ, then the pairs of strategies (x, y) and (x', y') violate the condition of strict competitiveness. ∎

Corollary 1. $a_{max} = a_{min} \quad \Leftrightarrow \quad b_{max} = b_{min}.$

If $a_{max} = a_{min}$ and $b_{max} = b_{min}$, then clearly B is an affine variant of A. If $a_{max} > a_{min}$ and $b_{max} > b_{min}$, we define the following affine variants of the matrices A and B.

$$A' = \frac{1}{a_{max} - a_{min}}[A - a_{min}U],$$

$$B' = \frac{1}{b_{max} - b_{min}}[B - b_{min}U].$$

Observe that all entries of A', B' are in $[0,1]$; in particular, both the value 0 and the value 1 appear as entries in both A' and B'. Moreover, $(A', -B')$ is a strictly competitive game. We show the following.

Lemma 2. $A' = B'$.

Proof. Suppose that $A' \neq B'$. By Lemma 1 and by rearranging the rows and columns of A' and B', we can assume without loss of generality that $A'_{11} = B'_{11} = 1$ and either $A'_{22} = B'_{22} = 0$ (case 1) or $A'_{12} = B'_{12} = 0$ (case 2). Let $D = B' - A'$ and let $|D_{rs}| = \max_{ij} |D_{ij}|$. For $0 \leq p \leq 1$, let $x(p),\ y(p)$ be probability vectors whose non-zero elements are:

- Case 1: $\quad x_1(p) = y_1(p) = p, \quad x_2(p) = y_2(p) = 1 - p$;
- Case 2: $\quad x_1(p) = 1, \quad y_1(p) = p,\ y_2(p) = 1 - p$.

Since $x(p)^{\mathrm{T}} A' y(p) = 0$ for $p = 0$ and $x(p)^{\mathrm{T}} A' y(p) = 1$ for $p = 1$, there exists $\bar{p}$ such that $x(\bar{p})^{\mathrm{T}} A' y(\bar{p}) = A'_{rs}$. Assuming $D_{rs} \neq 0$, we have $0 < \bar{p} < 1$. Since the game is strictly competitive, we have that $x(\bar{p})^{\mathrm{T}} B' y(\bar{p}) = B'_{rs}$. If this weren't the case, then by taking x' to be the pure strategy r and y' the pure strategy s we would obtain a contradiction to the strict competitiveness of the game by considering the pairs of mixed strategies $(x(\bar{p}), y(\bar{p}))$ and (x', y').

Given the above we have $x(\bar{p})^{\mathrm{T}} (A' + D) y(\bar{p}) = B'_{rs}$, which implies that $x(\bar{p})^{\mathrm{T}} D y(\bar{p}) = B'_{rs} - A'_{rs} = D_{rs}$. Noting that $D_{11} = 0$, $x_1(\bar{p}) \cdot y_1(\bar{p}) > 0$, $|D_{rs}| = \max_{ij} |D_{ij}|$, and that $x(\bar{p})^{\mathrm{T}} D y(\bar{p})$ is a weighted average of the elements in D, we can't have $x(\bar{p})^{\mathrm{T}} D y(\bar{p}) = D_{rs}$. Thus $D = 0$, implying $A' = B'$. ∎

Since A', B' are affine variants of A, B, this completes the proof of Theorem 1.

3 An Algebraic Proof

For any matrix A we consider the polynomial $p_A(z) = x^{\mathrm{T}} A y - x'^{\mathrm{T}} A y'$, where by z we denote the vector of variables x, y, x', y'. The hypothesis then states that $p_A(z)$ and $p_B(z)$ always have the same sign.

First note that, as polynomials, p_A and p_B are irreducible (there is no way to factor them without getting extra terms involving both primed and unprimed variables). Consider now the polynomial p_{A+B}, and consider a z^* such that $p_{A+B}(z^*) = 0$. It is easy to see that such a z^* exists. We claim that also $p_A(z^*) = 0$ — otherwise, $p_A(z^*)$ and $p_B(z^*) = p_{A+B}(z^*) - p_A(z^*)$ would have opposite signs.

We conclude that the roots of the irreducible polynomial $p_{A+B}(z)$ are a subset of the roots of the irreducible polynomial $p_A(z)$. It follows from Hilbert's Nullstellensatz [2] that $p_A(z)$ is a multiple of $p_{A+B}(z)$ (where we used that p_A is irreducible); [5] since p_A is irreducible, a constant multiple. Therefore, $p_A(z)$ and

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[5] Strictly speaking, the application of the Nullstellensatz requires that there is a non-singular point for the two polynomials with neighborhoods isomorphic to balls, but this is easy, if a little technical, to see—any point in the interior will do.

$p_B(z)$ are multiples of one another, and thus positive multiples.

Now, it is easy to see that $p_C(z) = p_D(z)$ iff C and D differ by a multiple of U (that is, the multiples of U comprise the kernel of the homomorphism from matrices A to polynomials p_A). We conclude that B is an affine variant of A, completing the proof.

References

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