Generalized Random Dot Product Models[1]

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Department of Mathematics

ACMS Talk
Dartmouth College
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[1] Joint work with Dan Rockmore

Abstract

The dot product model was defined in 1998 as a combinatorial method for efficiently representing graphs. More recently, randomized versions of this model have been shown to generate networks with small world properties and a vector embedding based on this model provides an effective framework for statistical inference for stochastic block models. In this talk I will describe a generalized version of the dot product model for networks with weighted edges focusing on the relationship between community structure and the vector embedding.

Outline

Philosophy

- Embedding data in mathematical objects

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- Discover interesting mathematical questions

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- Embedding data in mathematical objects
- Use mathematical properties to analyze data
- Discover interesting mathematical questions
- Make more efficient use of all of the data

Generative Models

- Noisy data

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- Null models

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- Mathematical tractability

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- Noisy data
- Null models
- Mathematical tractability
- Example networks

Dot Product Graphs

Definition (Dot Product Graph)

G is a dot product graph of dimension d if there exists a map $f : V(G) \to \mathbb{R}^d$ such that $(i,j) \in E(G)$ if and only if $\langle f(i), f(j) \rangle > 1$.

[2] C. Fiduccia, E. Scheinerman, A. Trenk, and J. Zito: Dot Product Representations of Graphs, Discrete Mathematics, 181, 1998, 113–138.

[3] R. Kang, L. Lovasz, T. Muller, and E. Scheinerman: Dot Product Representations of Planar Graphs, Electronic Journal of Combinatorics, 18, (2011), 1–14.

[4] Sphere and Dot Product Representations of Graphs: Discrete Computational Geometry, 47, (2012), 548–568.

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- Initial work: Fiduccia et al. (1998)[2]
- Planar graphs: Kang et al. (2011)[3]
- NP–Hard: Kang and Muller (2012)[4]
- $\frac{n}{2}$ critical graphs: Li and Chang (2014)[5]

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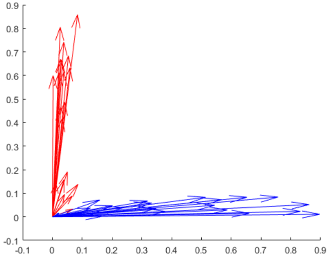
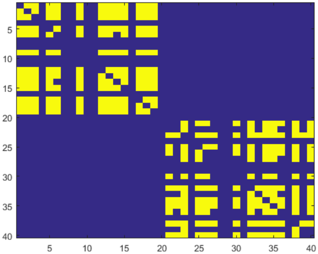
- Angle – Community assignment
- Magnitude – Centrality

Angle – Community Assignment

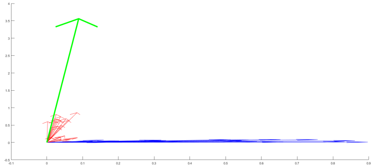


(a) Vectors



(b) Graph

Magnitude – Centrality



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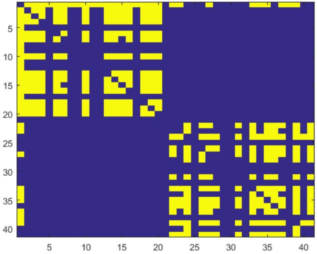

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Network Properties

- Initial work: Kraetzel et al. (2005)[6]
- General distributions: Young and Sceinerman (2007)[7]

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- Small world networks
  - Clustering
  - Small diameter
  - Degree distribution

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Statistical Applications

- Inverse problem: Scheinerman and Tucker (2010)[8]
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- Spectral Embedding and Statistics: Priebe Lab (2012–present)
  - Adjacency embedding[9]
  - Hypothesis testing[10]
  - Limit theorems[11]

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- Statistical results for WSBM[12]

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Example: Gaussian Edge Weights

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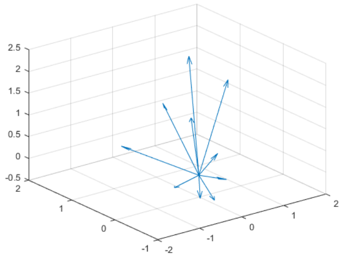
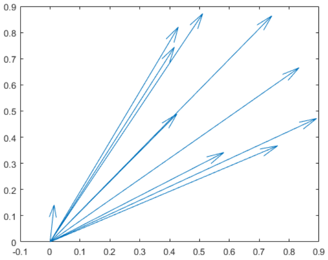
2) Choose $d_\mu = 3$ and $d_{sigma^2} = 2$
3) Take W_μ to be be independently normal in each component with mean 0 and variance 1 and W_{σ^2} to be uniform on $[0,1] \times [0,1]$.

Example: Gaussian Edge Weights

4)



(e) Mean Vectors



(f) Variance Vectors

Figure: Draws from W_{μ} and W_{σ^2}

WRDPM

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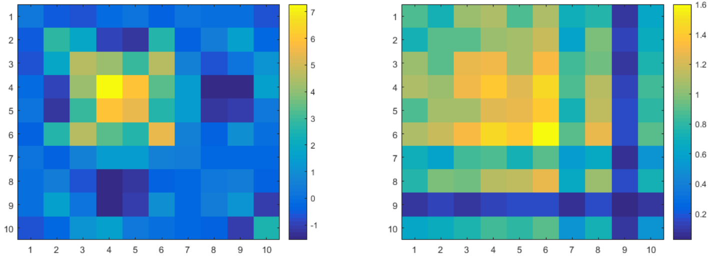
5)



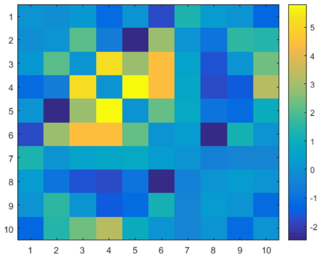
(a) Mean Dot Products (b) Variance Dot Products

Figure: Dot products for the vectors drawn in step 4.

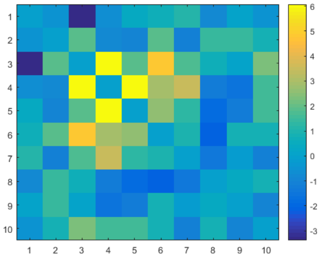
Dartmouth

Example: Gaussian Edge Weights

5)



(a) Sample Graph



(b) Sample Graph

Figure: Once the dot products are computed we can draw graphs from the distributions determined by the vectors.

Example: Uniform Noise

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0) Take P to be the Poisson distribution with parameters λ.
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2) Choose $d_\lambda = 3$
3) let Y be a normal random variable with mean 0 and variance .1 and take W_λ to be be defined by:

$$W_\lambda = \begin{cases} e_1 + Ye_1 + Ye_2 + Ye_3 & \frac{1}{3} \\ e_2 + Ye_1 + Ye_2 + Ye_3 & \frac{1}{3} \\ e_3 + Ye_1 + Ye_2 + Ye_3 & \frac{1}{3} \end{cases}$$

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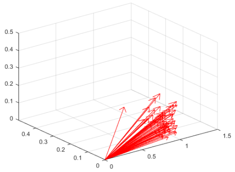
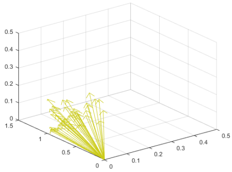
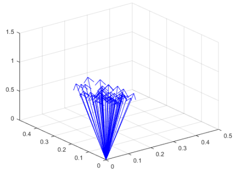


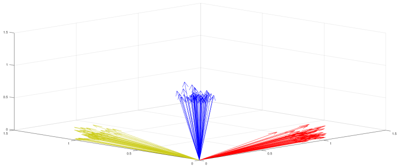
(a) Community 1 Vectors



(b) Community 2 Vectors



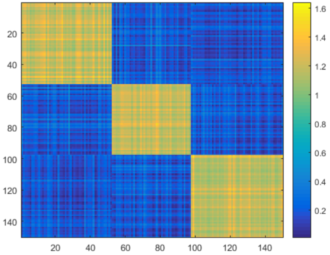
(c) Community 3 Vectors



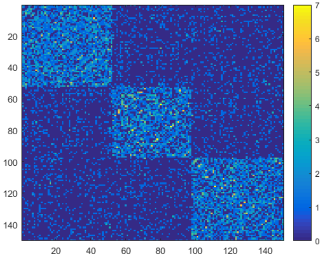
(d) All Vectors

Figure

Example: Uniform Noise



(a) Dot Products



(b) WRDPM Network

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WRDPM

Example: Multiresolution Communities

0) Take P to be the Poisson distribution with parameters λ.
1) Select $n = 150$.
2) Choose $d_\lambda = 3$
3) Let X be an exponential random variable with exponent 2, and take W_λ to be be defined by:

$$V_\lambda = \begin{cases} Xe_1 + Ye_2 + Ye_3 & \frac{1}{3} \\ Xe_2 + Ye_1 + Ye_3 & \frac{1}{3} \\ Xe_3 + Ye_1 + Ye_2 & \frac{1}{3} \end{cases}$$

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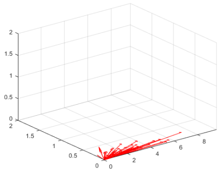
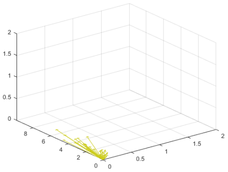
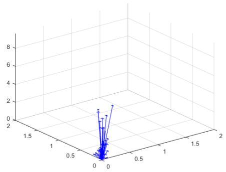


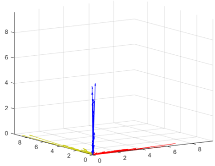
(c) Community 1 Vectors



(d) Community 2 Vectors



(e) Community 3 Vectors



(f) All Vectors

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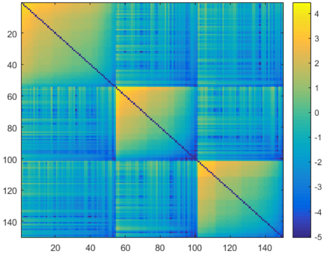
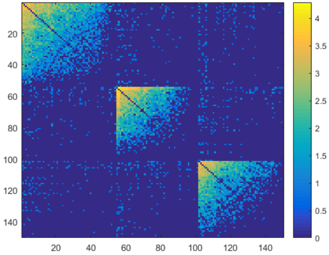


(g) Dot Products



(h) WRDPM Network

Edge Parameterized Models

Theorem

Let n be a fixed positive integer. For each pair (i,j) with $1 \leq i < j \leq n$ let $a_{i,j} = a_{j,i} \in \mathbb{R}$. Then there exist n real numbers $a_{\ell,\ell}$ for $1 \leq \ell \leq n$ such that the matrix $A_{i,j} = a_{i,j}$ is positive definite.

Proof.

Let the $a_{i,j}$ be selected arbitrarily. For $1 \leq \ell \leq n$ choose $a_{\ell,\ell} \in \mathbb{R}$ so that $a_{\ell,\ell} > \sum_{j \neq \ell} |a_{j,\ell}|$. Form a matrix A with $A_{i,j} = a_{i,j}$. This is a real symmetric matrix and so by the spectral theorem A has real eigenvalues. Applying Gershgorin's Circle Theorem to A gives that the eigenvalues of A lie in the closed disks centered at $a_{\ell,\ell}$ with radius $\sum_{j \neq \ell} |a_{j,\ell}|$. Intersecting these disks with the real line gives that the eigenvalues of A must lie in $\bigcup_{\ell=1}^{n} \left[a_{\ell,\ell} - \sum_{j \neq \ell} |a_{j,\ell}|, a_{\ell,\ell} + \sum_{j \neq \ell} |a_{j,\ell}|\right] \subseteq \mathbb{R}^{+}$. Thus, all eigenvalues of A are positive and A is positive definite. $\square$

Edge Parameterized Models

Corollary

Any generative network model, on a fixed number of nodes n, where the edge weight between each pair of nodes is drawn independently from a fixed probability distribution, possibly with different parameters for each pair, can be realized under the WRDPN.

Proof.

Let P be the k–parameter distribution from which the edge weights are drawn and for $1 \leq i \leq k$ let $a^i_{j,\ell} = a^i_{\ell,j}$ be the value of the ith parameter between nodes j and ℓ. Applying Theorem 1 to the collection $a^i_{j,\ell} = a^i_{\ell,j}$ gives a positive definite matrix A^i. Thus, there exists an $n \times n$ matrix X^i such that $(X^i)^T X^i = A$.

To form the WRDPM that matches the given generative model we take $d_i = n$ for all $1 \leq i \leq k$ and to each node $1 \leq j \leq n$ assign the collection of vectors given by the jth columns of the X^i for $1 \leq i \leq k$. $\square$

Examples

- Erdos–Renyi
  - Single vector for W
  - Simplest null model

[13] J. Ranola, S. Ahn, M. Sehl, D. Smith, and K. Lange: *A Poisson Model for random multigraphs*, Bioinformatics, 26, (2010), 2004–2011.

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  - Expected degree distribution
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- WSBM
  - Finite W
  - Community structure
  - Inference

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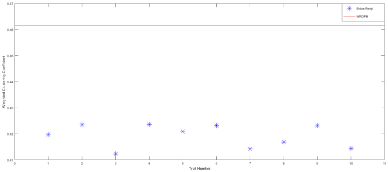
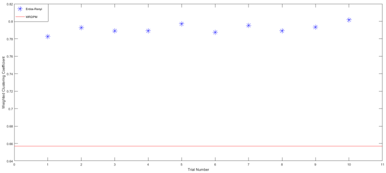
Weighted Clustering Coefficient



(i) Assortative Null Model



(j) Multiresolution Null Model

Network Properties

- Generative properties for weighted networks are more complex

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  - Clustering
  - Small diameter
- $k-$skeleton

Methodology

Want to find a collection of $d \times n$ vectors $\{X_i\}$ in order to approximate the entries of $A_{i,j}$ by $\langle X_i, X_j \rangle$. Equivalently, $X^T X \approx A$.

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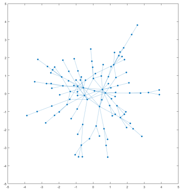
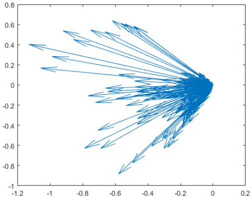
- Positive semi–definite approximation
- Extra degrees of freedom along diagonal
- Introduce a diagonal term
- Alternating, iterative optimization[14]

[14] E. Scheinerman and K. Tucker: *Modeling graphs using dot product representations*, Computational Statistics, 25, (2010), 1–16.

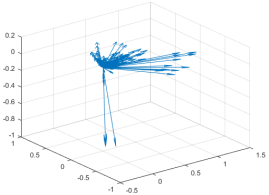
Unweighted Collaboration Network



(k) Collaboration Network



(l) Unweighted 2–Embedding

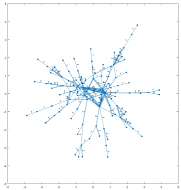


(m) Unweighted 3–Embedding

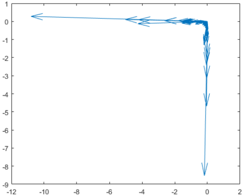
15

[15] V. Batagelj and A. Mrvar: *Pajek datasets*, (2006), URL: http://vlado.fmf.uni-lj.si/pub/networks/data/.

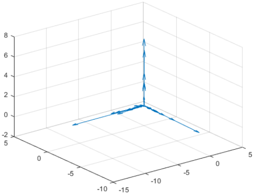
Weighted Collaboration Network



(n) Collaboration Network



(o) Weighted 2–Embedding



(p) Weighted 3–Embedding

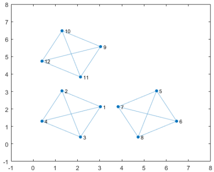
[16]

[16] V. Batagelj and A. Mrvar: *Pajek datasets*, (2006), URL: http://vlado.fmf.uni-lj.si/pub/networks/data/.

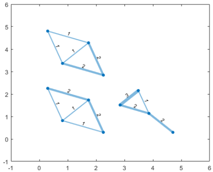
Interpretability

- Community structure: For most networks with assortative community structure, the approximation algorithm prioritizes separating distinct communities into nearly orthogonal components. Thus, the choice of dimension heavily influences the community representation.
- Centrality: Nodes that connect communities are assigned to longer vectors. This is related to betweeness centrality. Since length depends on community structure this is also affected by the choice of dimension.

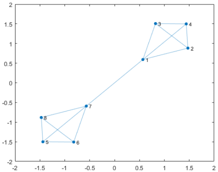
Examples



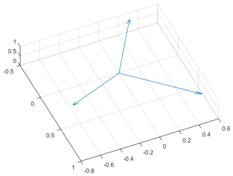
(q) Disjoint Cliques



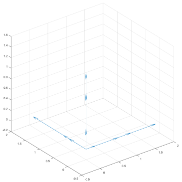
(r) Weighted Clusters



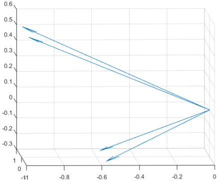
(s) Connecting Clusters



(t) Embedding of (q)



(u) Embedding of (r)



(v) Embedding of (s)

Dimension Selection

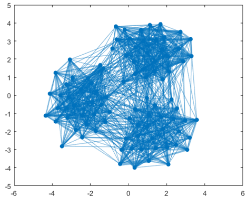
Since the dimension of the embedding is intrinsically related to the realized community structure it is natural to try and make use of this relationship to determine the right choice of d. Motivated by the case of disjoint communities, where if we have an effective, normalized embedding we should have

$$\langle X_i, X_j \rangle = \begin{cases} 1 & \text{i and j belong to the same community} \\ 0 & \text{i and j belong to different communities} \end{cases}$$

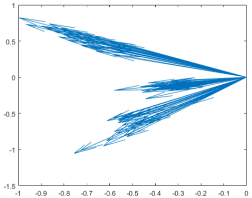
Thus, the sum of intra–community dot products should be $\sum_{i=1}^{\ell} \binom{z_\ell}{2}$. Similarly, the sum of the inter–community dot products should be 0. we define a stress function s depending on the community assignments after embedding.

$$s(d) = \sum_{i=1}^{d} \binom{z_i}{2} - \mathrm{s_{intra}}(d) + \mathrm{s_{inter}}(d)$$

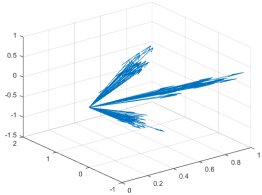
Dimension Example



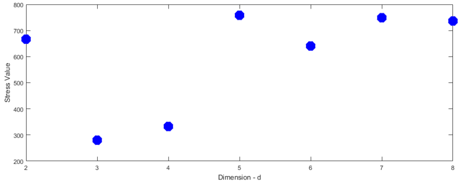
(w) Weighted Network



(x) 2-Embedding



(y) 3-Embedding



(z) Stress Function

Dartmouth

Coauthorship Revisited

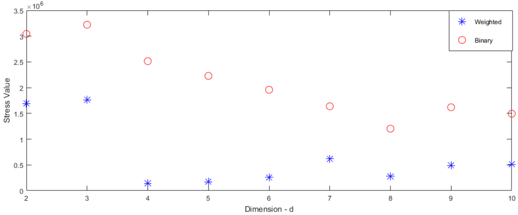
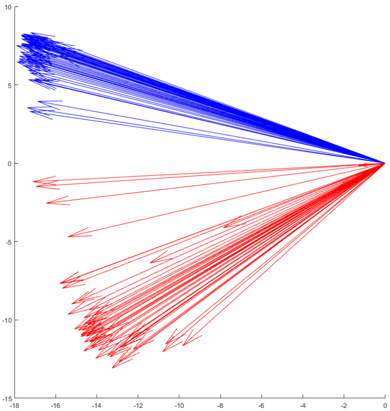


Figure: Comparison of stress values for the computational geometry coauthorship network between the weighted and unweighted realizations. The weighted embedding significantly outperforms the binarized model.

Voting Data



J. Lewis and K. Poole: *Roll Call Data*, voteview.com/dwnl.html.

Multiplex Networks and Multigraphs

- Frequently studied as aggregate objects[17]
- If the layers are independent these should be binomial
- Survey data and social networks
- No unbiased estimators[18]
- However, edge data is sparse and has large number of observations
- Synthetic examples
- Karnataka Villages data

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[17] D. Taylor, R. Caceres, and P. Mucha: *Detectability of small communities in multilayer and temporal networks: Eigenvector localization, layer aggregation, and time series discretization*, ArXiv 1609.04376, 1–14.

[18] A. DasGupta and H. Rubin: Estimation of binomial parameters when both n,p are unknown, Journal of Statistical Planning and Inference, 130, (2005), 391-404.

Multiplex Networks and Timeseries

- Estimating parameters for the edge weights requires more than a single sample for multivariate distributions
    - Block models
    - Multiplex networks
    - Time series data
- Fits into a broader program of robust network models for time series data
- Correlation Networks
- World Trade Web

Summary

- Network–theoretic analysis of the (W)RDPM
- Generalizes many previously studied models
- Natural interpretation of vector properties
- Dimension selection
- Current work:
    - Multiplex networks
    - Time series
    - Null models
    - Manifold properties

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That's all...

Thank You!