Generalized Random Dot Product Models\textsuperscript{1}

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\textsuperscript{1}Joint work with Dan Rockmore
Abstract

The dot product model was defined in 1998 as a combinatorial method for efficiently representing graphs. More recently, randomized versions of this model have been shown to generate networks with small world properties and a vector embedding based on this model provides an effective framework for statistical inference for stochastic block models. In this talk I will describe a generalized version of the dot product model for networks with weighted edges focusing on the relationship between community structure and the vector embedding.
Outline

1 Introduction
2 Dot Product Models
3 WRDPM
4 Special Cases

5 Generative Structure
6 Inference
7 Current Work
8 Conclusion
Philosophy

- Embedding data in mathematical objects
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- Use mathematical properties to analyze data
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- Discover interesting mathematical questions
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- Use mathematical properties to analyze data
- Discover interesting mathematical questions
- Make more efficient use of all of the data
Generative Models

- Noisy data
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- Null models
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- Mathematical tractability
Generative Models

- Noisy data
- Null models
- Mathematical tractability
- Example networks
Dot Product Graphs

**Definition (Dot Product Graph)**

$G$ is a dot product graph of dimension $d$ if there exists a map $f : V(G) \to \mathbb{R}^d$ such that $(i, j) \in E(G)$ if and only if $\langle f(i), f(j) \rangle > 1$.

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\(f : V(G) \rightarrow \mathbb{R}^d\) such that \((i, j) \in E(G)\) if and only if \(\langle f(i), f(j) \rangle > 1\).

- Initial work: Fiduccia et al. (1998)\(^2\)
- Planar graphs: Kang et al. (2011)\(^3\)
- NP–Hard: Kang and Muller (2012)\(^4\)
- \(n/2\) critical graphs: Li and Chang (2014)\(^5\)


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(RDPM 5): Form an adjacency matrix, $A$, form a network with $A_{j,\ell}$ drawn from Bernoulli($\langle X_j, X_\ell \rangle$) for $j \neq \ell$ and $A_{j,j} = 0$ for all $1 \leq j \leq n$. 
Interpretations

• Since each node is associated to a vector, it is natural to try and interpret the properties of the node from the vector
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\[ \langle x, y \rangle = \| x \| \cdot \| y \| \cos(x, y) \]

- Angle – Community assignment
- Magnitude – Centrality
Angle – Community Assignment

(a) Vectors

(b) Graph
Magnitude – Centrality

(c) Vectors

(d) Graph
Network Properties

• Initial work: Kraetzel et al. (2005)\textsuperscript{6}
• General distributions: Young and Scheinerman (2007)\textsuperscript{7}


Network Properties

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- General distributions: Young and Scheinerman (2007)\(^7\)
- Small world networks
  - Clustering
  - Small diameter
  - Degree distribution

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Statistical Applications

- Inverse problem: Scheinerman and Tucker (2010)\textsuperscript{8}
  - Iterative SVD for approximating $A_{i,j} = \langle X_i, X_j \rangle$
  - Angular k-means


Statistical Applications

- Inverse problem: Scheinerman and Tucker (2010)$^8$
  - Iterative SVD for approximating $A_{i,j} = \langle X_i, X_j \rangle$
  - Angular k-means
  - Adjacency embedding$^9$
  - Hypothesis testing$^{10}$
  - Limit theorems$^{11}$

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Why Generalize?

- Weighted Data

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• Multiplex Networks
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- Missing aspects from previous attempts (mostly network related)
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- Missing aspects from previous attempts (mostly network related)
- Statistical results for WSBM\textsuperscript{12}

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(WRDPM 4): For each node, $1 \leq j \leq n$, select $k$ vectors $1 \leq i \leq k$ (one from each parameter space), $X_i^j \in \mathbb{R}^{d_i}$, according to distribution $W_i$. 


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**WRDPM 4):** For each node, $1 \leq j \leq n$, select $k$ vectors $1 \leq i \leq k$ (one from each parameter space), $X_i^j \in \mathbb{R}^{d_i}$, according to distribution $W_i$.

**WRDPM 5):** Finally, construct a weighted adjacency matrix, $A$, for the network, with $A_{j,\ell}$ drawn according to $P(\langle X_1^\ell, X_1^j \rangle, \langle X_2^\ell, X_2^j \rangle, \ldots, \langle X_k^\ell, X_k^j \rangle)$ for $j > \ell$, $A_{j,\ell} = A_{\ell,j}$ for $j > \ell$ and $A_{j,j} = 0$ for all $1 \leq j \leq n$. 
Example: Gaussian Edge Weights

0) Take $P$ to be the normal distribution with parameters $\mu$ and $\sigma^2$
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Example: Gaussian Edge Weights

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1) Select $n = 10$.
2) Choose $d_\mu = 3$ and $d_{\sigma^2} = 2$
3) Take $W_\mu$ to be independently normal in each component with mean 0 and variance 1 and $W_{\sigma^2}$ to be uniform on $[0, 1] \times [0, 1]$. 
Example: Gaussian Edge Weights

Figure: Draws from $W_\mu$ and $W_{\sigma^2}$
Example: Gaussian Edge Weights

5)

(a) Mean Dot Products

(b) Variance Dot Products

Figure: Dot products for the vectors drawn in step 4.
Example: Gaussian Edge Weights

5)

(a) Sample Graph

(b) Sample Graph

Figure: Once the dot products are computed we can draw graphs from the distributions determined by the vectors.
Example: Uniform Noise

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2) Choose $d_\lambda = 3$
3) let $Y$ be a normal random variable with mean 0 and variance .1 and take $W_\lambda$ to be defined by:

$$W_\lambda = \begin{cases} 
  e_1 + Ye_1 + Ye_2 + Ye_3 & \frac{1}{3} \\
  e_2 + Ye_1 + Ye_2 + Ye_3 & \frac{1}{3} \\
  e_3 + Ye_1 + Ye_2 + Ye_3 & \frac{1}{3}
\end{cases}$$
Example: Uniform Noise

(a) Community 1 Vectors  (b) Community 2 Vectors  (c) Community 3 Vectors

(d) All Vectors

Figure
Example: Uniform Noise

(a) Dot Products

(b) WRDPM Network
Example: Multiresolution Communities

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3) Let $X$ be an exponential random variable with exponent 2, and take $W_\lambda$ to be defined by:

$$V_\lambda = \begin{cases} 
X e_1 + Y e_2 + Y e_3 & \frac{1}{3} \\
X e_2 + Y e_1 + Y e_3 & \frac{1}{3} \\
X e_3 + Y e_1 + Y e_2 & \frac{1}{3}
\end{cases}$$
Example: Multiresolution Communities

(c) Community 1 Vectors  (d) Community 2 Vectors  (e) Community 3 Vectors

(f) All Vectors
Example: Multiresolution Communities

(g) Dot Products  

(h) WRDPM Network
Theorem

Let $n$ be a fixed positive integer. For each pair $(i, j)$ with $1 \leq i < j \leq n$ let $a_{i,j} = a_{j,i} \in \mathbb{R}$. Then there exist $n$ real numbers $a_{\ell,\ell}$ for $1 \leq \ell \leq n$ such that the matrix $A_{i,j} = a_{i,j}$ is positive definite.

Proof.

Let the $a_{i,j}$ be selected arbitrarily. For $1 \leq \ell \leq n$ choose $a_{\ell,\ell} \in \mathbb{R}$ so that $a_{\ell,\ell} > \sum_{j \neq \ell} |a_{j,\ell}|$. Form a matrix $A$ with $A_{i,j} = a_{i,j}$. This is a real symmetric matrix and so by the spectral theorem $A$ has real eigenvalues. Applying Gershgorin’s Circle Theorem to $A$ gives that the eigenvalues of $A$ lie in the closed disks centered at $a_{\ell,\ell}$ with radius $\sum_{j \neq \ell} |a_{j,\ell}|$. Intersecting these disks with the real line gives that the eigenvalues of $A$ must lie in $\bigcup_{\ell=1}^{n} \left[ a_{\ell,\ell} - \sum_{j \neq \ell} |a_{j,\ell}|, a_{\ell,\ell} + \sum_{j \neq \ell} |a_{j,\ell}| \right] \subseteq \mathbb{R}^+$. Thus, all eigenvalues of $A$ are positive and $A$ is positive definite.
Edge Parameterized Models

Corollary

Any generative network model, on a fixed number of nodes $n$, where the edge weight between each pair of nodes is drawn independently from a fixed probability distribution, possibly with different parameters for each pair, can be realized under the WRDPN.

Proof.

Let $P$ be the $k$–parameter distribution from which the edge weights are drawn and for $1 \leq i \leq k$ let $a_{j,\ell}^i = a_{\ell,j}^i$ be the value of the $i$th parameter between nodes $j$ and $\ell$. Applying Theorem 1 to the collection $a_{j,\ell}^i = a_{\ell,j}^i$ gives a positive definite matrix $A^i$. Thus, there exists an $n \times n$ matrix $X^i$ such that $(X^i)^T X^i = A$.

To form the WRDPM that matches the given generative model we take $d_i = n$ for all $1 \leq i \leq k$ and to each node $1 \leq j \leq n$ assign the collection of vectors given by the $j$th columns of the $X^i$ for $1 \leq i \leq k$. \qed
Examples

- Erdos–Renyi
  - Single vector for $W$
  - Simplest null model

\[\text{\cite{Ranola2010}}\]
Examples

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  - Simplest null model
- Chung–Lu
  - One–dimensional model
  - Expected degree distribution
  - Poisson version: Ranola et al. (2010)\textsuperscript{13}

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- WSBM
  - Finite $W$
  - Community structure
  - Inference

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Weighted Clustering Coefficient

(i) Assortative Null Model

(j) Multiresolution Null Model
Generative properties for weighted networks are more complex.
Network Properties

- Generative properties for weighted networks are more complex
- What is a weighted small world network?
• Generative properties for weighted networks are more complex
• What is a weighted small world network?
• Standard small world + weights
Generative properties for weighted networks are more complex
What is a weighted small world network?
Standard small world + weights
Reparameterize the Bernoulli model: $1 - e^{\langle X_i, X_j \rangle}$
- Clustering
- Small diameter
• Generative properties for weighted networks are more complex
• What is a weighted small world network?
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• Reparameterize the Bernoulli model: \( 1 - e^{\langle X_i, X_j \rangle} \)
  • Clustering
  • Small diameter
• \( k \)-skeleton
Want to find a collection of $d \times n$ vectors $\{X_i\}$ in order to approximate the entries of $A_{i,j}$ by $\langle X_i, X_j \rangle$. Equivalently, $X^T X \approx A$. 

\cite{Scheinerman2010}
Methodology

Want to find a collection of $d \times n$ vectors $\{X_i\}$ in order to approximate the entries of $A_{i,j}$ by $\langle X_i, X_j \rangle$. Equivalently, $X^T X \approx A$.

- Positive semi–definite approximation
- Extra degrees of freedom along diagonal
- Introduce a diagonal term
- Alternating, iterative optimization\(^{14}\)

Inference

Unweighted Collaboration Network

(k) Collaboration Network  (l) Unweighted 2–Embedding  (m) Unweighted 3–Embedding

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Weighted Collaboration Network

(n) Collaboration Network

(o) Weighted 2–Embedding

(p) Weighted 3–Embedding

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Interpretability

- Community structure: For most networks with assortative community structure, the approximation algorithm prioritizes separating distinct communities into nearly orthogonal components. Thus, the choice of dimension heavily influences the community representation.

- Centrality: Nodes that connect communities are assigned to longer vectors. This is related to betweenness centrality. Since length depends on community structure this is also affected by the choice of dimension.
Examples

(q) Disjoint Cliques

(r) Weighted Clusters

(s) Connecting Clusters

(t) Embedding of (q)

(u) Embedding of (r)

(v) Embedding of (s)
Dimension Selection

Since the dimension of the embedding is intrinsically related to the realized community structure it is natural to try and make use of this relationship to determine the right choice of $d$. Motivated by the case of disjoint communities, where if we have an effective, normalized embedding we should have

$$\langle X_i, X_j \rangle = \begin{cases} 1 & \text{i and j belong to the same community} \\ 0 & \text{i and j belong to different communities} \end{cases}$$

Thus, the sum of intra–community dot products should be $\sum_{i=1}^{\ell} \binom{z_i}{2}$. Similarly, the sum of the inter–community dot products should be 0. We define a stress function $s$ depending on the community assignments after embedding.

$$s(d) = \sum_{i=1}^{d} \binom{z_i}{2} - s_{\text{intra}}(d) + s_{\text{inter}}(d)$$
Figure: Comparison of WRDPN embeddings of a weighted network (a) as the dimension of the embedding varies. As expected, the minimum value occurs at $d = 3$, matching the correct structure.

(w) Weighted Network  (x) 2-Embedding  (y) 3-Embedding

(z) Stress Function
Figure: Comparison of stress values for the computational geometry coauthorship network between the weighted and unweighted realizations. The weighted embedding significantly outperforms the binarized model.
J. Lewis and K. Poole: *Roll Call Data*, voteview.com/dwnl.html.
Multiplex Networks and Multigraphs

- Frequently studied as aggregate objects\(^{17}\)
- If the layers are independent these should be binomial
- Survey data and social networks
- No unbiased estimators\(^{18}\)
- However, edge data is sparse and has large number of observations
- Synthetic examples
- Karnataka Villages data


\(^{18}\)A. DasGupta and H. Rubin: Estimation of binomial parameters when both \(n,p\) are unknown, *Journal of Statistical Planning and Inference*, 130, (2005), 391-404.
Multiplex Networks and Timeseries

- Estimating parameters for the edge weights requires more than a single sample for multivariate distributions
  - Block models
  - Multiplex networks
  - Time series data
- Fits into a broader program of robust network models for time series data
- Correlation Networks
- World Trade Web
Summary

- Network-theoretic analysis of the (W)RDPM
- Generalizes many previously studied models
- Natural interpretation of vector properties
- Dimension selection
- Current work:
  - Multiplex networks
  - Time series
  - Null models
  - Manifold properties
References


That’s all...

Thank You!