Multiplex Structure on the World Trade Web

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Abstract

Analyzing dynamics on graphs leads to some of the most important tools and invariants in complex networks. In this talk we present an algebraic method for extending these techniques to multiplex networks, in terms of an operator that connects the endogenous and exogenous dynamics on the graph. As a case study, we present a multiplex analysis of the World Trade Web using our operator.

Outline

World Trade Web

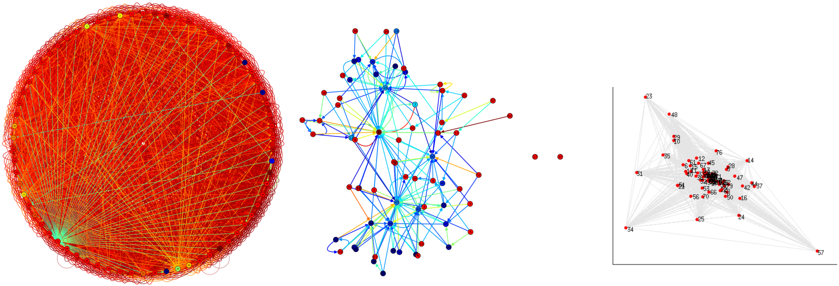


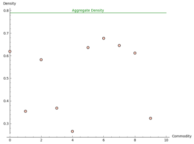
Figure : Visual Realizations of the 2000 WTW

Commodity Definitions

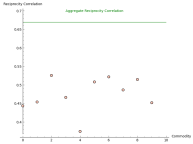
Layer	Description	Volume	% Total	Transitivity
0	Food and live animals	291554437	5.1	.82
1	Beverages and tobacco	48046852	0.9	.67
2	Crude materials	188946835	3.3	.79
3	Mineral fuels	565811660	10.0	.62
4	Animal and vegetable oils	14578671	0.3	.64
5	Chemicals	535703156	9.5	.83
6	Manufactured Goods	790582194	13.9	.87
7	Machinery	2387828874	42.1	.85
8	Miscellaneous manufacturing	736642890	13.0	.83
9	Other commodities	107685024	1.9	.56
All	Aggregate Trade	5667380593	100	.93

Table : Commodity information for the WTW

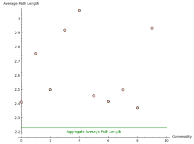
Layer Metrics



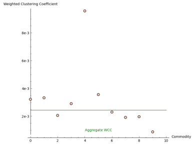
Density



Reciprocity



Path Length



Clustering

Figure : Comparisons of standard network metrics between the aggregate WTW and the individual commodity networks

Degree Distributions

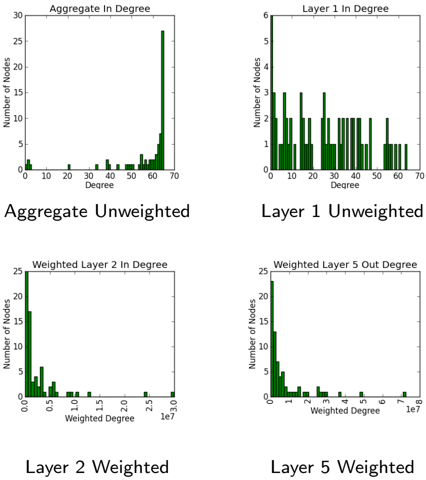


Figure : Representative Degree Distributions

Multiplex Definition

Definition

A multiplex is a collection of graphs all defined on the same node set.

The motivations for studying these objects are mostly practical:

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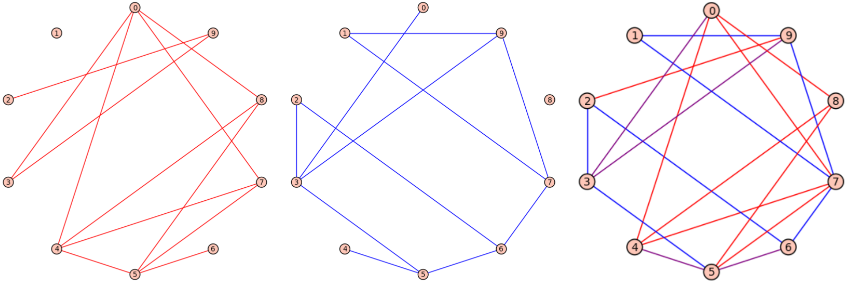
The motivations for studying these objects are mostly practical:

- Trade networks
- Social networks
- Neural networks
- Anonymity networks

Toy Multiplex



(a) Layer 1 (b) Layer 2 (c) Multiplex

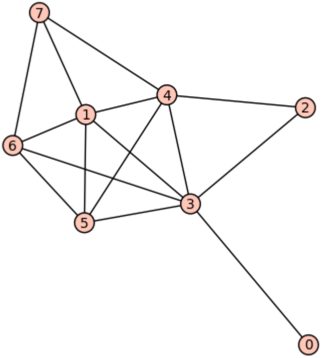
Spectral Graph Theory

Spectral graph theory studies invariants of graphs using the spectral structure of associated matrices.

Process:

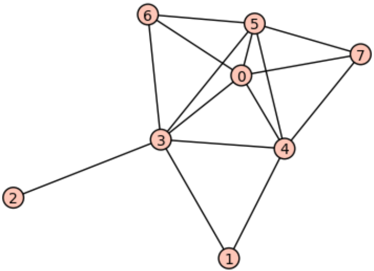
- Structural Representation

Adjacency Matrix



$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Laplacian



$$L = \begin{bmatrix} 5 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 2 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 6 & -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & -1 & 5 & -1 & 0 & -1 \\ -1 & 0 & 0 & -1 & -1 & 5 & -1 & -1 \\ -1 & 0 & 0 & -1 & 0 & -1 & 3 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & 3 \end{bmatrix}$$

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  - Flows across edges
  - Normalized (AD^{-1}) leads to random walks

Dynamics on Networks

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- Adjacency Matrix
  - $v_i = \sum_{i\sim j} v_j = \sum_j A_{i,j} v_j$
  - Flows across edges
  - Normalized (AD^{-1}) leads to random walks
- Laplacian
  - Heat flow
  - Isoperimetric clustering
  - Random walks $AD^{-1} = D^{-\frac{1}{2}}(I - D^{-\frac{1}{2}} L D^{-\frac{1}{2}}) D^{\frac{1}{2}}$

Spectral Graph Theory

Spectral graph theory studies invariants of graphs using the spectral structure of associated matrices.

Process:

- Structural Representation
- Dynamical Interpretation
- Spectral Analysis

Clustering

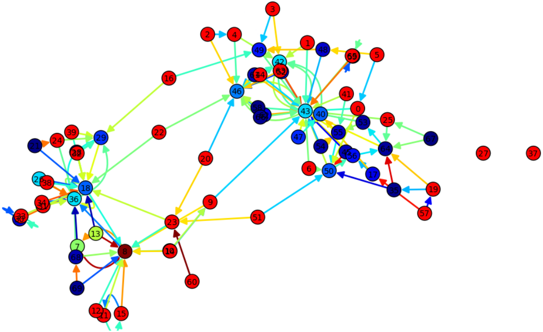


Figure : 2-out subgraph of the Manufactured Goods trade network

Clustering

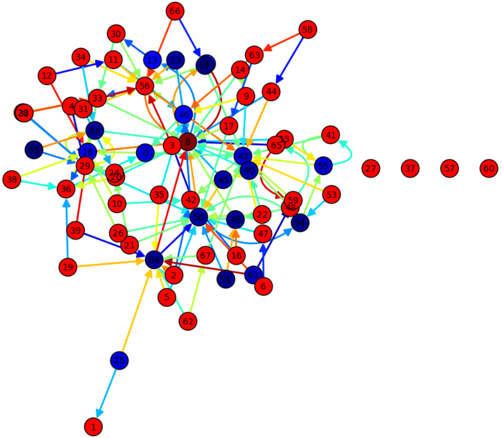
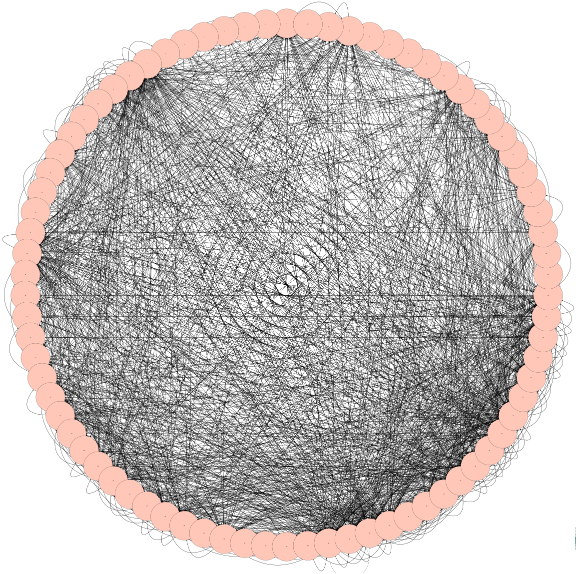


Figure : 2-out subgraph of the Other Commodities trade network

Multiplex Dynamics



Motivation

Many of these structures have intrinsic dynamics that distinguish between connections between distinct nodes and connections between copies of the same node. Early approaches to studying graph problems in this context tried to address this problem from a structural perspective[1] (summing matrices or adding edges between copies). These approaches tend to distort the metrics of interest by conflating the intra and inter relationships.

[1] S. Gomez, A. Diaz-Guilera, J. Gomez-Gardenes, C.J. Perez-Vicente, Y. Moreno, and A. Arenas: *Diffusion Dynamics on Multiplex Networks*, Physical Review Letters, 110, (2013).

Algebraic Approach

Instead of trying to add new structural components we connect the dynamics using a collection of scaled orthogonal projections. To each node, we associate a projection operator P_n that gathers the information stored at each node and proportionally redistributes it among the copies. This allows us to respect the independence of the endogenous dynamics.

Linear Case

In this linear case this is particularly convenient. Given a collection of operators D_i on our layers, this is equivalent to constructing the new operator:

$$M = \begin{bmatrix} \alpha_{1,1}C_1D_1 & \alpha_{1,2}C_1D_2 & \cdots & \alpha_{1,k}C_1D_k \\ \alpha_{2,1}C_2D_1 & \alpha_{2,2}C_2D_2 & \cdots & \alpha_{2,k}C_2D_k \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{k,1}C_kD_1 & \alpha_{k,2}C_kD_2 & \cdots & \alpha_{k,k}C_kD_k \end{bmatrix}$$

where the $C_i = \mathrm{diag}(c_{i,1}, c_{i,2} \ldots, c_{i,\ell})$ represent the coefficients for the node projections with the condition that $\sum_j c_{i,j} = 1$ for all i.

Preserved Properties

The types of questions we are interested in depend on the initial properties of the dynamics, like positive definiteness or stochasticity. In order to interpret the results about our operator it must share these properties.

Theorem (Condensed)

If the original dynamics are
$\{$*Irreducible, Primitive, Stochastic, Positive(negative) (semi–)Definite*$\}$
then M *is*
$\{$*Irreducible, Primitive, Stochastic, Positive(negative) (semi–)Definite*$\}$.

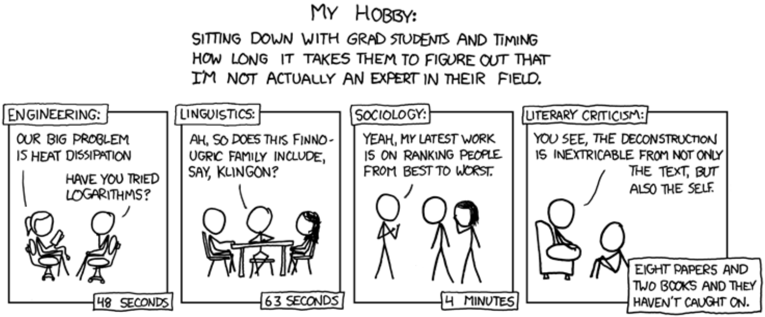
Laplacian Bounds

Theorem

If the flows between layers are equidistributed and the individual dynamics are the associated network Laplacians we have the following eigenvalue bounds:

- *Fiedler Value:* $\max_i(\lambda_f^i) \leq \lambda_f \leq \min_i(\lambda_1^i) + \sum_{j \neq \ell} \lambda_f^j$,
- *Leading Value:* $\min_i(\lambda_1^i) \leq \lambda_1 \leq \sum_i \lambda_1^i$,
- *Synchronization Stability:* $\frac{\min_i(\lambda_1^i)}{\min_i(\lambda_1^i) + \sum_{j \neq \ell} \lambda_f^j} \leq G_{ss} \leq \frac{\sum_i \lambda_1^i}{\max_i(\lambda_f^i)}$.

Centrality

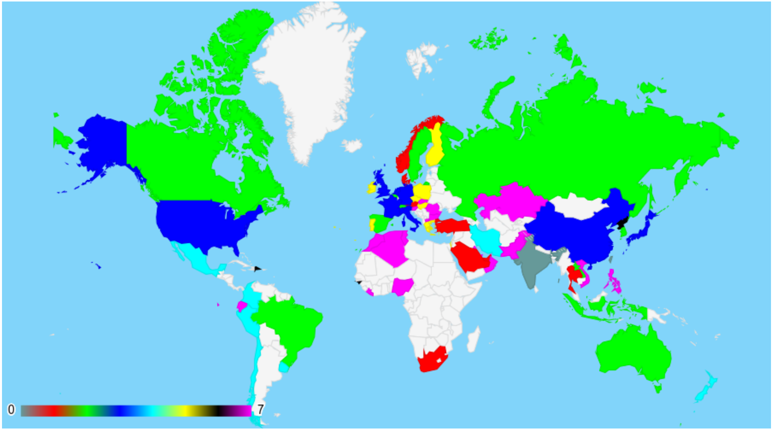

MY HOBBY:
SITTING DOWN WITH GRAD STUDENTS AND TIMING HOW LONG IT TAKES THEM TO FIGURE OUT THAT I'M NOT ACTUALLY AN EXPERT IN THEIR FIELD.
ENGINEERING:
OUR BIG PROBLEM IS HEAT DISSIPATION
HAVE YOU TRIED LOGARITHMS?
48 SECONDS
LINGUISTICS:
AH, SO DOES THIS FINNO-UGRIC FAMILY INCLUDE, SAY, KLINGON?
63 SECONDS
SOCIOLOGY:
YEAH, MY LATEST WORK IS ON RANKING PEOPLE FROM BEST TO WORST.
4 MINUTES
LITERARY CRITICISM:
YOU SEE, THE DECONSTRUCTION IS INEXTRICABLE FROM NOT ONLY THE TEXT, BUT ALSO THE SELF.
EIGHT PAPERS AND TWO BOOKS AND THEY HAVEN'T CAUGHT ON.

Multiplex Centrality

	Equal		In Strength		Out Strength		Layer Strength	
Rank	Layer	Country	Layer	Country	Layer	Country	Layer	Country
1	All	USA	7	Japan	7	USA	7	USA
2	All	Canada	7	USA	7	Canada	7	Japan
3	All	Japan	7	Mexico	7	Mexico	7	Canada
4	All	China	7	Canada	7	Japan	7	Mexico
5	All	Mexico	7	Germany	7	China	7	China
6	All	Germany	8	China	3	Japan	7	Germany
7	All	UK	7	S. Korea	7	Germany	6	USA
8	All	France	7	China	8	USA	8	USA
9	All	S. Korea	7	Laos	8	Japan	6	Japan
10	All	Italy	8	USA	7	Laos	7	S. Korea

Table : Multiplex Centrality Leaders

Community Detection



Conclusions

- The aggregate WTW is not very representative of the underlying economic structure
- Better information about the WTW can be obtained by viewing it as a multiplex of **directed** networks
- Our multiplex operator allows these structures to be analyzed using standard network theoretic tools as well as providing a general framework for multiplex analysis

References

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- R. Feenstra, R. Lipsey, H. Deng, A. Ma, H. Mo: *World Trade Flows: 1962–2000*, Working Paper 11040, NBER.
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That's all...

Thank You!