2016 AMC 10A PROBLEM #17

CROSSROADS ACADEMY
AMC PREPARATION

1. Let N be a positive multiple of 5. One red ball and N green balls are arranged in a line in random order. Let $P(N)$ be the probability that at least $\frac{3}{5}$ of the green balls are on the same side of the red ball. What is the sum of the digits of the least value of N such that $P(N) < \frac{321}{400}$?

I would approach this problem by first thinking about what the possible arrangements look like. For any fixed value of N we can think about laying out the green balls in a row and inserting the red ball in one of $N-1$ gaps between green balls or before or after all of the green balls. This means that the denominator of $P(N)$ is $N+1$.

To get a handle on the numerator we can look at two separate (and symmetric) cases, where $\frac{3}{5}$ of the green balls lie to the left (right) of the of the red ball. In order to have $\frac{3}{5}$ of the green balls to the left of the red ball we can place the red ball in any position from directly after the $\frac{3N}{5}$th ball to the position after the Nth ball. This gives $N+1-\frac{3N}{5}=\frac{2N}{5}+1$ positions. By symmetry there are another $\frac{2N}{5}+1$ positions where $\frac{3}{5}$ of the green balls lie to the right of the of the red ball for a total of $\frac{4N}{5}+2$.

This gives us a formula to work with:

$$P(N)=\frac{\frac{4N}{5}+2}{N+1}$$

This is a decreasing function of N since as N increases the denominator gets larger by 1 and the numerator gets larger by $\frac{4}{5}$. This tells us that if we can solve $P(x)=\frac{321}{400}$ then the smallest integer value of N that satisfies the strict inequality in the problem will be $\lfloor x+1\rfloor$. Luckily, the algebra is not so hard:

$$\begin{aligned} P(x) &= \frac{321}{400} \\ \frac{\frac{4x}{5}+2}{x+1} &= \frac{321}{400} \\ 320x+800 &= 321x+321 \\ 479 &= x \end{aligned}$$

Thus, $N=480$ and the sum of the digits is $4+8+0=12$ which is choice (A).

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