2017 AMC 10 PROBLEM #?

CROSSROADS ACADEMY
AMC PREPARATION

1. How many three digit numbers have the property that their digits can be permuted to make a number divisible by 11?

This is a pretty neat problem. There is probably a more clever way to do this than I am going to present here but hopefully this approach will at least make sense. The key idea is that we are going to split the three digit multiples of 11 into classes by how many legal permutations that they have. To start, we compute that there are $\frac{990-110}{11}+1=81$ total multiples of 11 to deal with.

If we represent three digit numbers by their digit representation $a(100)+b(10)+c$, then the divisibility rule for 11 means that we have to have $a-b+c\equiv 0 \pmod{11}$ or equivalently $a+c\equiv c \pmod{11}$. We will use this fact over and over again in this solution. There are four separate cases that we have to worry about, mostly depending on the zeros, since when we consider permutations we aren't allowed to have a zero in the hundreds place. In each case, we will count the number of multiples of 11 that satisfy the condition and then count how many legal permutations of those multiples count towards our final answer.

Case 1: The first case is when $a=b$ and $c=0$. There are 9 possible choices for $a=b$, so this case contains 9 multiples of 11. Each multiple counts twice towards our final answer: $aa0$ and $a0a$ giving a total contribution of 18 from case 1.

Case 2: The second cases is when $a=c$ and $b\neq 0$. There are 8 choices for $a=c$, since we can't have $a=0$ or $a=c=5$ to make our $\pmod{11}$ equivalence work. Each of these multiples corresponds to 3 permutations since it is determined by which place b occupies: bac, abc, and acb. Thus, case 2 contributes 24 to our final answer.

Case 3: The third case has $a\neq c$ and $b=0$. There are 8 choices for a that each give a unique corresponding choice for c. For example, if $a=4$ then $c=7$. We can't choose $a=1$ since there are no multiples of 11 between 100 and 109. Each multiple of this form has four legitimate permutations $a0c$, $ac0$, $ca0$, and $c0a$ but we are over–counting by a factor of two, since the pairs like $a=4$ and $c=7$ also occur as $a=7$ and $c=4$. To count correctly we add $\frac{8}{2}\cdot 4=16$ to our total tally.

Case 4: The remaining multiples of 11 have $a\neq b\neq c$ and none of the digits are zero. After subtracting off the previous cases, there are $81-9-8-8=56$ remaining multiples of 11 each of which can be permuted in 6 ways. However, as in Case 3 we are over–counting our multiples by a factor of 2 since if abc is a multiple of 11 then so is cba by our formula. That means that this case contributes a total of $\frac{56}{2}\cdot 6=168$ to our sum.

Putting these cases together we get $18+24+16+126=226$ numbers as the final answer.

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Just for fun, here is the full list of numbers: 101, 110, 112, 121, 123, 132, 134, 139, 143, 145, 148, 154, 156, 157, 165, 166, 167, 175, 176, 178, 184, 187, 189, 193, 198, 202, 209, 211, 213, 220, 224, 231, 235, 242, 246, 249, 253, 257, 258, 264, 267, 268, 275, 276, 279, 285, 286, 290, 294, 297, 303, 308, 312, 314, 319, 321, 325, 330, 336, 341, 347, 352, 358, 359, 363, 368, 369, 374, 377, 380, 385, 386, 391, 395, 396, 404, 407, 413, 415, 418, 422, 426, 429, 431, 437, 440, 448, 451, 459, 462, 469, 470, 473, 478, 481, 484, 487, 492, 495, 496, 505, 506, 514, 516, 517, 523, 527, 528, 532, 538, 539, 541, 549, 550, 560, 561, 571, 572, 579, 582, 583, 588, 593, 594, 597, 605, 606, 615, 616, 617, 624, 627, 628, 633, 638, 639, 642, 649, 650, 651, 660, 661, 671, 672, 682, 683, 689, 693, 694, 698, 704, 707, 715, 716, 718, 725, 726, 729, 734, 737, 740, 743, 748, 751, 752, 759, 761, 762, 770, 773, 781, 784, 792, 795, 799, 803, 808, 814, 817, 819, 825, 826, 830, 835, 836, 841, 844, 847, 852, 853, 858, 862, 863, 869, 871, 874, 880, 885, 891, 896, 902, 909, 913, 918, 920, 924, 927, 931, 935, 936, 942, 945, 946, 953, 954, 957, 963, 964, 968, 972, 975, 979, 981, 986, 990, 997.