2015 STATE SOLUTIONS

CROSSROADS ACADEMY
MATHCOUNTS PREPARATION

1. Problem 26

Problem. *Each of the 25 cells in a five–by–five grid of squares is filled with a 0, 1, or 2 in such a way that the numbers written in neighboring cells differ from the number on that cell by 1. Two cells are considered neighbors if they share a side. How many different arrangements are possible?*

Solution 1. *The key idea here is that if any square contains a 0 or 2 then all of its neighboring squares must be 1's. Imagine coloring the board like a chessboard with the upper right square black: the previous sentence means that either all of the white squares or all of the black squares must be ones. For each of the non 1 squares we have two choices that are entirely independent so we can count them all at once. If the black squares have ones then there are* $2^{12} = 4096$ *ways to fill in the white squares with either 0's or 2's. If the white squares have ones then there are* $2^{13} = 8192$ *ways to fill in the black squares with 0's or 2's. Summing these gives the answer:* $4096 + 8192 = 12,288$.

2. Problem 30

Problem. *How many ways are there to arrange the digits 1 through 9 in a* 3×3 *grid, such that the numbers are increasing from left to right in each row and increasing top to bottom in each column?*

Solution 2. *This type of problem – filling in a grid with increasing integers in both directions – is known as a standard young tableaux. These are incredibly useful objects that have applications all over mathematics. The number of standard young tableaux is given by the "hook–length" formula, which doesn't have a particularly intuitive interpretation – it requires some fairly advanced techniques to prove – but it fairly easy to write down:*

For a young diagram with n blocks we look at each block in position (i, j) *individually and compute* $b(i, j)$*: the number of blocks that are directly below or directly to the right of that block including itself. For example in our* 3×3 *grid for the upper left block we have 2 to the right, 2 below, and 1 for itself for a total of 5, while the center block has 1 to the right, 1 below, and 1 for itself for a total of 3. We then multiply these all together and divide* $n!$ *by the product:*

$$\frac{n!}{\prod_{i,j} b(i,j)}$$

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For this problem there are 9 blocks and the $b(i,j)$ are easy to compute:

$$\begin{aligned}
b(1,1) &= 5\\
b(1,2) &= 4\\
b(2,1) &= 4\\
b(1,3) &= 3\\
b(2,2) &= 3\\
b(3,1) &= 3\\
b(3,2) &= 2\\
b(2,3) &= 2\\
b(3,3) &= 1
\end{aligned}$$

Then, applying the hook length formula gives:

$$\frac{9!}{5\cdot 4\cdot 4\cdot 3\cdot 3\cdot 3\cdot 2\cdot 2\cdot 1} = \frac{362880}{8640}$$

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