HARMONIC MEAN PROBLEM

CROSSROADS ACADEMY
MATHCOUNTS PREPARATION

1. Problem Statement

A robot paces back and forth along a desert trail for all eternity. Besides the robot, the trail is entirely empty except for an infinite stream of westward bound, evenly–spaced stagecoaches each moving at a fixed constant rate of speed. When the robot is walking west it is passed by a stagecoach every 12 minutes. When it is walking east it is passed by a stagecoach every 4 minutes. Eventually, the robots knee joints rust and it is forced to remain in place. How often do stagecoaches pass the robot after it gets stuck?

2. Solution

Let d be the distance between the stagecoaches, s be the speed of the stagecoaches, and r be the walking speed of the robot. To apply the rate formula, we observe that when the robot is walking east we can start counting our time immediately when the robot is passed by a stagecoach. Then, $d = (s + r)4$. Similarly, considering the robot walking west gives $d = (s - r)12$. Setting these two equations equal to each other gives:

$$(s+r)4 = (s-r)12 \tag{2.1}$$
$$16r = 8s \tag{2.2}$$
$$2r = s \tag{2.3}$$

This unfortunately does not seem like enough information to figure out the actual problem we are trying to solve. However, there is another way we can combine the equations. Imagine that the robot turns around from traveling east to traveling west just as it is passed by a stagecoach. Consider the 16 minutes from when the robot was last passed by a stagecoach (while he was traveling east) until he is next passed by a stage coach (traveling west). Rewriting our rate equations using $t = \frac{d}{r}$, we get the following from summing the transformed versions of the two original equations:

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\begin{aligned}
16 &= \frac{d}{s+r} + \frac{d}{s-r} && (2.4)\\
16 &= \frac{d(s-r+s+r)}{(s+r)(s-r)} && (2.5)\\
16 &= \frac{2ds}{(s-r)(s+r)} && (2.6)\\
16 &= \frac{8ds}{3s^2} && (2.7)\\
16 &= \frac{8d}{3s} && (2.8)\\
6 &= \frac{d}{s} && (2.9)\\
6s &= d && (2.10)
\end{aligned}
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Where equation (2.7) follows by substitution from the equality we derived in (2.3). This final equation, $6s = d$, gives the answer to problem, since d is the original distance and s is the original robot speed, so each stagecoach covers the distance in 6 minutes, which is how frequently the stationary robot is passed. □

3. Notes

The approach needed for this problem probably seems a little unintuitive. One of the real problems with the rate formula is that two of the three formulations are expressed in terms of quotients instead of products. This means that our normal algebraic approaches won't always work. In this particular case the motivation from our solutions comes from the idea of the harmonic mean. The harmonic mean of two numbers a and b is defined to be $\dfrac{2}{\frac{1}{a}+\frac{1}{b}}$. It has several geometric interpretations, relating triangles and their inscribed and circumscribed circles.

The harmonic mean is probably most frequently appears in competitions in the context of the RMS-**HM**–GM–AM–QM inequality. The second most common version is the one that is useful in this problem: the harmonic mean is the average speed over an entire trip for a person who travels a fixed distance d at speed x and then the same distance d at speed y. Then, the total time is equal to $\frac{d}{x}+\frac{d}{y}$ which should look familiar if you worked through the solution in the previous section... The average speed is the total distance divided by the total time: $\dfrac{2d}{\frac{d}{x}+\frac{d}{y}} = \dfrac{2}{\frac{1}{x}+\frac{1}{y}}$ which hopefully also looks familiar.

Department of Mathematics, Dartmouth College
E-mail address: ddeford@math.dartmouth.edu