AIME PREPARATION

1. Warmup Problems

(1) Let $P(n)$ and $S(n)$ denote the product and the sum, respectively, of the digits of the integer n. For example, $P(23) = 6$ and $S(23) = 5$. Suppose N is a two-digit number such that $N = P(N) + S(N)$. What is the units digit of N?

(2) The mean of three numbers is 10 more than the least of the numbers and 15 less than the greatest. The median of the three numbers is 5. What is their sum?

(3) A telephone number has the form ABC-DEF-GHIJ, where each letter represents a different digit. The digits in each part of the number are in decreasing order; that is, $A > B > C$, $D > E > F$, and $G > H > I > J$. Furthermore, D, E, and F are consecutive even digits; G, H, I, and J are consecutive odd digits; and $A+B+C = 9$. Find A.

(4) A box contains exactly five chips, three red and two white. Chips are randomly removed one at a time without replacement until all the red chips are drawn or all the white chips are drawn. What is the probability that the last chip drawn is white?

(5) How many positive integers not exceeding 2017 are multiples of 3 or 4 but not 5?

(6) An insect lives on the surface of a regular tetrahedron with edges of length 1. It wishes to travel on the surface of the tetrahedron from the midpoint of one edge to the midpoint of the opposite edge. What is the length of the shortest such trip? (Note: Two edges of a tetrahedron are opposite if they have no common endpoint.)

(7) A spider has one sock and one shoe for each of its eight legs. In how many different orders can the spider put on its socks and shoes, assuming that, on each leg, the sock must be put on before the shoe?

(8) The polynomial $P(x) = x^3 + ax^2 + bx + c$ has the property that the mean of its zeros, the product of its zeros, and the sum of its coefficients are all equal. If the y-intercept of the graph of $y = P(x)$ is 2, what is b?

(9) Points $A = (3, 9)$, $B = (1, 1)$, $C = (5, 3)$, and $D = (a, b)$ lie in the first quadrant and are the vertices of quadrilateral $ABCD$. The quadrilateral formed by joining the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ is a square. What is the sum of the coordinates of point D?

(10) Consider sequences of positive real numbers of the form $x, 2000, y, \ldots$ in which every term after the first is 1 less than the product of its two immediate neighbors. For how many different values of x does the term 2001 appear somewhere in the sequence?

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2. Other Warmup Problems

(1) Point P is 9 units from the center of a circle of radius 15. How many different chords of the circle contain P and have integer lengths?

(2) Jack and Jill run 10 km. They start at the same point, run 5 km up a hill, and reurn to the starting point by the same route. Jack has a 10 minute head start and runs at the rate of 15 km/hr uphill and 20 km/hr downhill. Jill runs 16 km/hr uphill and 22 km/hr downhill. How far from the top of the hill are they when they pass each other going in opposite directions (in km)?

(3) A positive integer N is a palindrome if the integer obtained by reversing the sequence of digits of N is equal to N. The year 1991 is the only year in the current century with the following 2 properties:

(a) It is a palindrome (b) It factors as a product of a 2-digit prime palindrome and a 3-digit prime palindrome.

How many years in the millenium between 1000 and 2000 have properties (a) and (b)?

(4) Triangle ABC has a right angle at $C, AC = 3$ and $BC = 4$. Triangle ABD has a right angle at A and $AD = 12$. Points C and D are on opposite sides of $\overline{AB}$. The line through D parallel to $\overline{AC}$ meets $\overline{CB}$ extended at E. If

$$\frac{DE}{DB} = \frac{m}{n},$$

where m and n are relatively prime positive integers, then $m + n =$

(5) If $ABCD$ is a 2×2 square, E is the midpoint of $\overline{AB}$,F is the midpoint of $\overline{BC}$,$\overline{AF}$ and $\overline{DE}$ intersect at I, and $\overline{BD}$ and $\overline{AF}$ intersect at H, then the area of quadrilateral $BEIH$ is

(6) Are there integers a and b such that $a^5b + 3$ and $ab^5 + 3$ are both perfect cubes of integers?