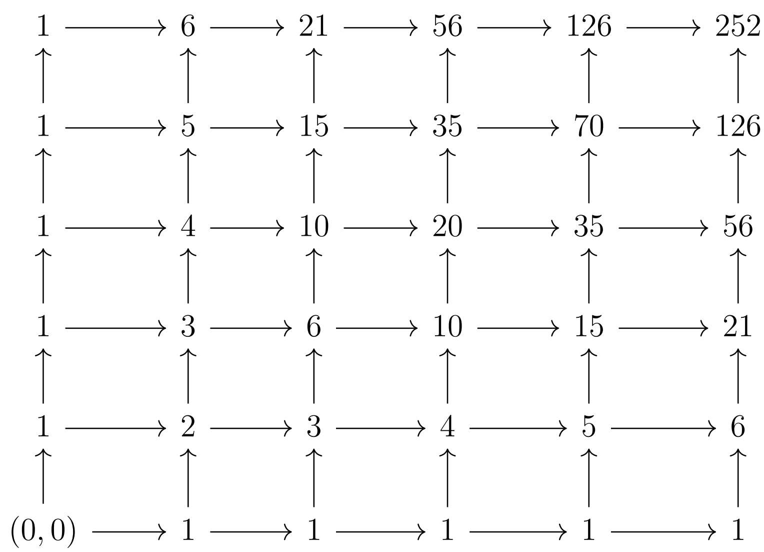
PATH COUNTING PROBLEMS

CROSSROADS ACADEMY
AMC PREPARATION

Problems about counting paths on a graph or a lattice are another common type of competition problem. These questions also have relations to many other kinds of counting problems like the Catalan numbers and graph theory.

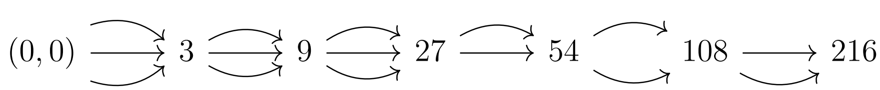
1. Path Counting Basics

The simplest problems deal with counting paths on a rectangular lattice. Usually the formulation is something like: "beginning at the origin, how many distinct paths are there to (m,n) using only upward and rightward steps of one unit each?" The answer to this question is given by a binomial coefficient $\binom{m+n}{m} = \binom{m+n}{n}$, we know that we have to take m steps to the right and n steps up. Choosing the positions of the m steps right (or equivalently the n steps up) uniquely determines one of the distinct paths. This leads to the following (hopefully familiar) answers for small values of m and n:



2. Graph Paths

Sometimes the problems are asked about less regular graphs than the grid. In these cases we need to use more clever methods to do the counting. Usually the best way to approach these problems is to split the points into smaller problems by focusing on intermediate points that all paths must pass through.



Date: September 20, 2016.

3. Games with Path Counting

For each of these games can you identify the player with a winning strategy for any choice of m and n?

3.1. **Evens or Odds.** Start with a marker at the origin of an $m \times n$ grid. The first player moves either one square up or to the right. The second player then also chooses to move up or right. The players alternate moves until the marker reaches the outer (top or right) edge of the board at point (p, q). If $p + q$ is odd player one wins, if $p + q$ is even player two wins.

3.2. **Path Chomp.** Start with a marker at the origin of an $m \times n$ grid. The first player moves as many squares in one (up or right) direction as they want. The players alternate moves until reaching point (m, n). The player who moves to (m, n) loses. Alternate version: each player may move as many squares as they want in both directions (e.g. Alice starts by moving up 2 and right 4 then Bob moves right 3 and up 1). Again, the player who moves to (m, n) loses.

3.3. **Road Construction.** Start with a marker at the origin of an $m \times n$ grid. One player is the driver and the other player is the construction worker. The driver starts by moving the marker one square any direction (up, down, left, or right as long as they don't leave the grid). Then the construction player erases one edge from the grid subject to two rules: First the removed edge can't be next to the vertex containing the marker and Second there must always be a clear path from the origin to (m, n). The players alternate until either the driver wins by reaching (m, n) or the construction player wins by making it impossible for the driver to make it. Alternate version: the construction worker deletes points instead of edges.

4. Warm–up Problems

(1) How many paths are there on a rectangular grid from $(0, 0)$ to $(7, 3)$?

(2) How many paths are there on a rectangular grid from $(0, 0)$ to $(7, 3)$ that go through the point $(2, 2)$?

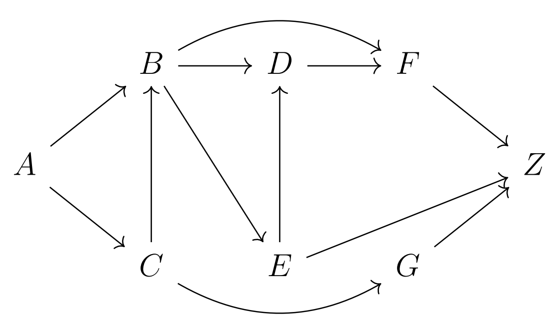
(3) How many paths are there on a rectangular grid from $(0, 0)$ to $(7, 3)$ that don't go through the point $(2, 2)$?

(4) How many paths are there on a rectangular grid from $(0, 0)$ to $(7, 3)$ going through either $(2, 2)$ or $(3, 3)$?

(5) How many paths are there on a rectangular grid from $(0, 0)$ to $(7, 3)$ going through both $(2, 2)$ and $(3, 3)$?

(6) Can you construct a graph that has 4 paths between point A and B? What about a graph with 12 paths between A and B? Can you construct a graph that has 5 paths between point A and B as well as 12 paths between A and C?

(7) Consider the graph below. How many paths exist from A to Z?



(8) Consider one–dimensional paths starting from the origin and only moving to the right but moving any number of steps at a time. How many paths are there from $(0, 0)$ to $(4, 0)$? What about to $(n, 0)$?

5. Intermediate Problems

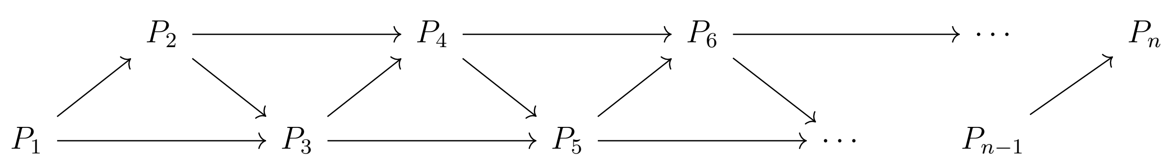
(1) Consider a knight on the lower–left square of a regular 8×8 chessboard. How many paths can the knight take to reach the upper–right square if he can only move up 2 right 1 or up 1 right 2? What if he is also allowed to move up 2 left 1 and up 1 and left 2?

(2) If a rook begins at the lower left square of a 2×4 chessboard, how many paths can he take to the upper right square if he can only move up or right? Can you find a formula for a general $2 \times n$ chessboard?

(3) If a queen begins at the lower left square of a 2×4 chessboard, how many paths can he take to the upper right square if she can only move up or right? Can you find a formula for a general $2 \times n$ chessboard?

(4) Consider paths moving only up or right on a 7×3 grid, beginning at the lower left square. What is the smallest positive integer that does not occur as a possible number of paths to a point?

(5) How many paths are there from P_1 to P_n in the following diagram:



(6) Imagine that a person starts at the origin and flips a coin, moving right every heads and up every tails. What is the probability that they reach the square $(3, 4)$. What about $(5, 2)$? Is there a general formula for reaching (m, n)?

(7) Imagine that a person starts at the origin and rolls two die, moving right the amount shown on the first die and up the amount on the second die. What is the probability that they reach the square $(3, 4)$. What about $(7, 7)$? Is there a general formula for reaching (m, n)?

6. Catalan and Delannoy Numbers

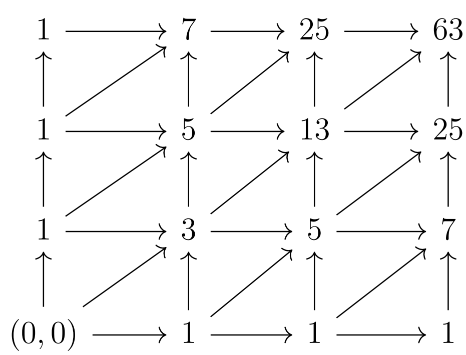
The Catalan numbers count the number of paths from $(0,0)$ to $(2n,n)$ that do not cross the line $y=x$ and are related to many interesting counting problems.

(1) Compute the first 4 Catalan numbers.

(2) Show that the Catalan numbers satisfy the formula $\frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}$.

(3) Show that the Catalan numbers count the number of ways to arrange n pairs of parentheses so that they are correctly matched.

(4) The Delannoy numbers count the number of paths from $(0,0)$ to (m,n) on the following grid with diagonals:



(5) Prove a Pascal–like identity for the Delannoy numbers

(6) Can you find a formula for the diagonal sums of the Delannoy numbers? Or even a second order recurrence relation? These are known as the Pell numbers (although they are usually not defined this way).

7. Advanced Problems

(1) Consider paths in the three dimensional grid starting at $(0, 0, 0)$ moving up, right, or in. How many paths are there to $(2, 3, 4)$?

(2) Consider paths in the three dimensional grid starting at $(0, 0, 0)$ moving up, right, or in. How many paths are there to $(2, 3, 4)$ where each step must remain on the 'outside' of the rectangular prism?

(3) Consider paths on the two dimensional grid where you can move up, down, left, or right. How many paths of length $2n$ start and end at the origin.

(4) Consider paths on the three dimensional grid where you can move up, down, in, out, left, or right. How many paths of length $2n$ start and end at the origin.

(5) A walk on a graph is a path that can repeat nodes and edges. How many walks of length 2 are there on a cycle with five vertices? Is there a formula for a cycle of n vertices? What about a path with n vertices?

(6) Draw a graph on 4 points with 7 paths between point A and point B.

(7) Draw a graph on 4 points with 10 paths between point A and point B that must go through point C.

(8) Draw a graph on any number of points that has 16 paths connecting point A and point B. What is the fewest number of edges that must be drawn to build such a graph? What is the most number of edges that can be drawn to build such a graph?