PLAYING CARD PROBLEMS

CROSSROADS ACADEMY
AMC PREPARATION

Problems using playing cards are some of the most common counting and probability questions that occur on the AMC. This sheet describes some of the basic facts and numbers about playing cards that are frequently used in problems.

1. Playing Card Vocabulary

A standard playing card deck consists of 52 cards, each of which is defined by a suit and a rank. There are four different suits: Hearts, Diamonds, Spades, and Clubs. The Hearts and Diamonds are red and the Spades and Clubs are black. There are 13 different ranks: 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, and Ace. The Jack, Queen, and King are collectively known as the "face" cards. Some decks also include two joker or wild cards for a total of 54.

There are a few standard 5–card poker hands that have special names associated to different collections of cards. In ascending order of rarity they are: one pair, two pairs, three of a kind, straight (5 sequential cards in a row like 7, 8, 9, 10, Jack), Flush (5 cards all of the same suit), full house (3 cards of a single rank and 2 cards of a different rank), four of a kind, and straight flush (5 sequential cards of the same suit).

2. Some Playing Card Numbers

- 52 cards in the deck
- 13 cards in each suit
- 4 cards of each rank
- 26 red cards
- 26 black cards
- 20 even numbered card
- 16 odd numbered cards
- 16 prime numbered cards
- 12 face cards
- 2,598,960 5 card poker hands
- Most counting problems assume that you begin with a shuffled deck, meaning the cards are placed in a random order. There are: 80,658,175,170,943,878,571,660,636,856,403,766,975,289,505,440,883,277,824,000,000,000,000 ways to shuffle the entire deck. In words this is 80 unvigintillion 658 vigintillion 175 novemdecillion 170 octodecillion 943 septendecillion 878 sexdecillion 571 quindecillion 660 quattuordecillion 636 tredecillion 856 duodecillion 403 undecillion 766 decillion 975 nonillion 289 octillion 505 septillion 440 sextillion 883 quintillion 277 quadrillion 824 trillion. This enormous number is sometimes the denominator of the probabilities that we want to compute :).

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3. Games with Playing Cards

3.1. **Math War!** This is a game for two or more players each using their own deck of cards. Each horizontal row is a game board. Each player draws the top card of their deck and places it in one of the available spaces in the chosen row. Continue until all of the spaces in the row are filled and then compute the value of the expression. Face cards count as zeroes and aces count as ones. The player with the highest value wins the round. Play best 4 out of 7 rounds. For the later boards red cards have negative values and black cards are positive.

3.2. **Cribbage Scoring.** In cribbage, cards with numbers on them have value equal to their number, aces ares ones, and all face cards count as tens. A cribbage hand consists of 5 cards and points are scored in the following ways:

- 2 points for each pair
- 1 point per card for each run of three or longer
- 2 points for each collection of cards whose values sum to 15
- 1 point per card for having four or five cards of the same suit

Draw four cards from the top of the deck. Compute the expected value of your hand based on the cards remaining in the deck. Draw the top card and compute the score of your hand. How does it compare to the expected value? In regular cribbage, you draw six cards and keep four of them, and then complete your hand by revealing the top card of the deck. Draw six cards and compute the expected value for each four card subset. Then reveal the top card. Which 4 card hand would have scored the most points.

3.3. **Concentration.** Lay out all of the cards in the deck face down in a grid. Each player takes turns revealing two cards. If the revealed cards are the same rank the player that revealed them removes them from the grid and keeps them face up in a separate pile. If the revealed cards are not the same rank they are placed face down again and the next player may reveal two cards. Once all of the cards are face up the player who collected the most pairs wins. For a more mathematical version instead of removing cards when they have the same rank, remove them if the sum of their values adds up to a perfect square.

4. Warm–up Problems

(1) What is the probability of selecting a heart from a shuffled deck of cards?

(2) What is the probability that the top card of a shuffled deck is red and a queen?

(3) What is the probability that the top card of a shuffled deck is red or a queen?

(4) What is the probability of drawing a five and then a two with replacement?

(5) What is the probability of drawing a five and then a two without replacement?

(6) What is the probability that one of the top two cards is a king?

(7) What is the probability that at least one of the top two cards is a king?

(8) What is the probability that at most one of the top two cards is a king?

(9) What is the probability that the top card of a shuffled deck is a black seven and the bottom card is a heart?

(10) What is the probability that the top card of a shuffled deck is a black seven and the bottom card is a spade?

5. Intermediate Problems

(1) What is the probability of selecting three red cards in a row with replacement? What about without replacement? How do these formulas change if we want to select n red cards in a row?

(2) What is the probability of selecting two cards of different suits with replacement? What about without replacement? What if we want to select two cards of different ranks instead?

(3) What is the probability that the top four cards of a deck are all face cards? What is the probability that at least three of the top four cards are face cards?

(4) How many 5 card poker hands have exactly three face cards? How many have at least three face cards?

(5) What is the probability that the top four cards of a shuffled deck are two pairs? What if we want one black pair and one red pair?

(6) What is the probability that the top 10 cards in a shuffled deck are red? What is the probability that the top k cards in a shuffled deck are red?

(7) For each of the eight poker hand types on the front page, compute the number of possible 5 card hands of that type.

6. Advanced Problems

(1) What is the probability that a 5 card poker hand contains the Ace of Spades? What is the probability that a 13 card hand contains the Queen of Spades? What is the probability that a k card hand has the six of diamonds?

(2) What is the probability that the top 10 cards of a shuffled deck alternate between red and black? What is the probability that the top 12 cards of a shuffled deck alternate between spade - heart - diamond - club and then repeat?

(3) What is the probability that a 5 card poker hand has no repeated ranks?

(4) What is the probability that a 5 card poker hand has exactly two cards of one suit and one card of each other suit?

(5) In the game of cribbage, aces are worth 1 point, numbered cards are worth their face value, and face cards are all worth 10 points. What is the expected value of the top card of a shuffled cribbage deck? What is the probability that the top two cards of a shuffled cribbage deck sum to 15?

(6) What is the probability that there are no 4's in the top half of a shuffled deck?

(7) What is the probability that the top four cards in a shuffled deck are consecutive cards of the same suit?

7. Challenge Problems

(1) Go back to an interesting previous problems and replace the number of cards with n. Can you find a general formula for these problems? What if you change the problems to be with replacement?

(2) How many 5 card hands contain at least one card of each suit? How about n card hands?

(3) How many 5 card hands contain no cards of exactly one suit? What about 5 card hands containing no cards of at least one suit? How about n card hands?

(4) Remove the aces from a standard deck and separate the remaining cards into four shuffled piles, one for each suit. Assign points to each rank as follows:

Rank	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Points	2	3	4	5	6	7	8	9	10	11	12	13

Compute the expected value of the sum of the four top cards. Now consider the following different assignment of points to ranks (where the black and red cards have separate values):

Black Rank	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Points	2	4	5	6	7	8	9	10	11	12	13	15
Red Rank	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Points	2	3	3	4	4	5	8	9	9	10	10	11

Compute the expected value of the sum of the four top cards using these new point values. Prove the stronger result that the probabilities of obtaining any particular sum of the top four cards are the same for both of these point systems. Can you find another set of (non–negative) point values with this property? Is there a set of points where each suit has a different set of point values with this property?

(5) Label the cards in the deck from 1–52. Consider continually repeating the following process: Flip over the top card of the deck. If the number of the top card is k, remove the top k cards from the deck and place them back on top of the deck in the opposite order. Prove that eventually the card labeled 1 will be on top of the deck.