PATH COUNTING

NEW HAMPSHIRE STATE TEAM
NATIONAL MATHCOUNTS PREPARATION

1. Introduction

Problems 3 and 4 in Section 7 of the path counting problems are fun examples of different ways to think about paths. The formula for problem 3 is actually pretty well behaved but the one for problem 4 is fairly messy, even though they both come from the same type of intuition.

These types of problems frequently appear in probability classes in the guise of random walks, where at every step the walker flips a fair coin to decide which direction to go. Then, we might ask for the probability that a walker beginning at the origin returns to the origin, or even for the expected value of the number of times the walker returns to the origin if he wanders infinitely. In order to get some intuition for this problem we will start with the one–dimensional case before moving on to the actual problems.

2. 1-d Paths

In the 1-d setting we are looking at paths on the number line, starting at 0, where we can step either to the right or left. For any natural number n, any path of length $2n$ only returns to the origin if there are exactly as many steps to the left as to the right. There are exactly $\binom{2n}{n}$ of these paths since we can "choose" the positions of the n left steps and fill in the remaining steps with rights.

Since there are 2^{2n} total possible paths of length $2n$ the probability that a random path ends up back at 0 is $\frac{\binom{2n}{n}}{2^{2n}}$.

3. 2-d Paths

We can approach the 2–d case with the same logic as the previous version, only now we must balance both the left/right steps and the up/down steps. That is, if we represent our path as a sequence of l, r, u, d steps we must have that the number of l is the same as the number of r and the number of u must be the same as the number of d but it doesn't make any difference what order the letters occur.

For any natural number n, we can determine a path of length $2n$ by choosing some number $k \leq n$ of steps to the left which forces us to have k steps to the right as well as $n-k$ steps up and $n-k$ steps down for $k+k+(n-k)+(n-k) = 2n$ total steps. To order the steps we can permute the $2n$ steps in $(2n)!$ ways but then we have to divide by $(k!)^2$ and $(n-k)!^2$ since the steps in the same direction are not distinct. This gives:

$$\sum_{k=0}^{n} \frac{(2n)!}{(k!)^2(n-k)!^2}$$

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total paths. We can actually simplify this a little further using a binomial identity which says that $\sum_{k=0}^{n}\binom{n}{k} = \binom{2n}{n}$[1]:

$$\begin{aligned}
\sum_{k=0}^{n} \frac{(2n)!}{(k!)^2(n-k)!^2} =& (2n!)\sum_{k=0}^{n} \frac{1}{(k!)^2(n-k)!^2} \\
=& (2n!)\sum_{k=0}^{n} \frac{1}{(k!)^2(n-k)!^2}\frac{(n!)^2}{(n!)^2} \\
=& \frac{(2n!)}{(n!)^2}\sum_{k=0}^{n} \left(\frac{(n!)}{(k!)(n-k)!}\right)^2 \\
=& \binom{2n}{n}\sum_{k=0}^{n} \binom{n}{k}^2 \\
=& \binom{2n}{n}^2
\end{aligned}$$

which is an even nicer formula.

4. 3-D Paths

For the 3–d case we need to balance left/right, up/down, and out/in steps. For a natural number n we can form a path of length $2n$ by choosing $k \leq n$ left (and right) steps together with $j \leq (n-k)$ up (and down) steps and $n-k-j$ out (and in) steps. Rearranging the steps as in the 2–d case we get the following formula:

$$\sum_{k+j\leq n} \frac{(2n)!}{(k!)^2(j!)^2((n-k-j)!)^2}$$

which unfortunately doesn't simplify as nicely.

[1]This identity has a clever proof based on counting the number of lattice paths from $(0,0)$ to (n,n) using only up and right steps in two different ways.