Èdouard Lucas:

The theory of recurrent sequences is an inexhaustible mine which contains all the properties of numbers; by calculating the successive terms of such sequences, decomposing them into their prime factors and seeking out by experimentation the laws of appearance and reproduction of the prime numbers, one can advance in a systematic manner the study of the properties of numbers and their application to all branches of mathematics.

Computational Challenges for Neutral Redistricting

Daryl DeFord

MIT – CSAIL
Geometric Data Processing Group

Graphics Seminar
MIT – CSAIL
March 13, 2019

Outline

Collaborators

- Prof. Justin Solomon — **MIT CSAIL**
- Lorenzo Najt — **Wisconsin Math**
- Prof. Moon Duchin — **Tufts Math**
- Hugo Lavenant — **Universitè Paris–Sud Math**
- Zachary Schutzman — **UPenn CIS**

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- Zachary Schutzman — **UPenn CIS**

- Voting Rights Data Institute
  - 52 undergraduate and graduate students
  - 6–8 week summer program
  - mggg.org
  - github.com/{gerrymandr,mggg,mggg-states}

MORAL #1:

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Computational Redistricting is NOT a solved problem!

MORAL #2:

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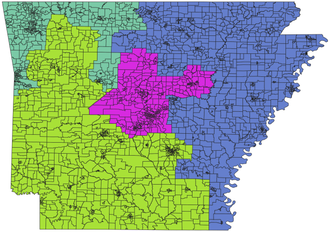
MORAL #2:

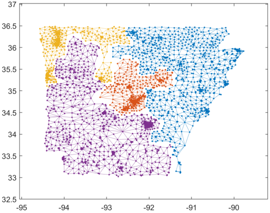
Computational Redistricting is NOT a solved problem!

tinyurl.com/gerryprojects

Political Partitioning

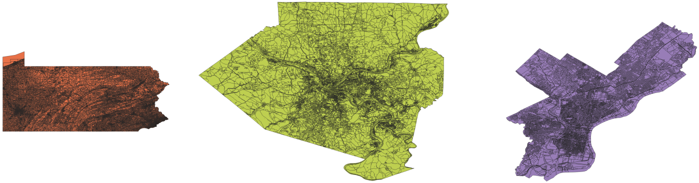


(a) Geography



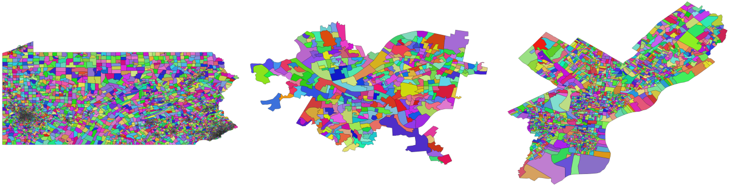
(b) Dual Graph

Blocks



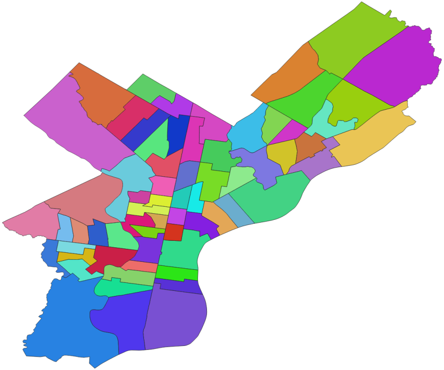
(a) Pennsylvania (b) Allegheny (c) Philadelphia

Precincts

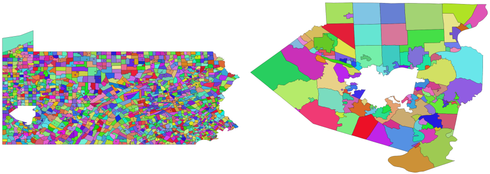


(a) Pennsylvania (b) Pittsburgh (c) Philadelphia

Wards

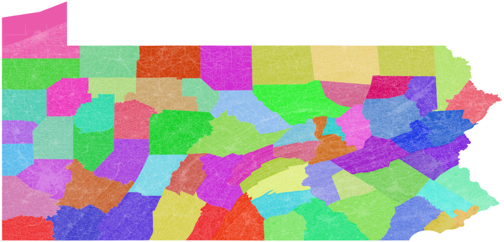


Municipalities

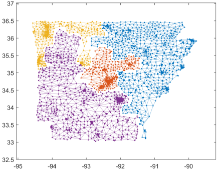


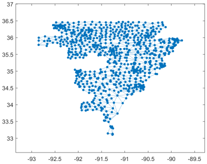
(a) Pennsylvania (b) Allegheny

Counties

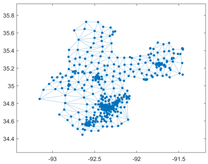


Arkansas Congressional Districts

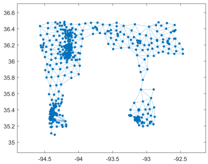




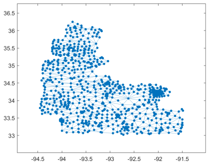
(b) District # 1



(c) District # 2



(d) District # 3



(e) District # 4

Permissible Districting Plans

We want to partition a given geography (graph), at a given scale, into k pieces, satisfying some constraints:

- Contiguity
- Population Balance
- Compactness
- Communities of Interest
- Municipal Boundaries
- Competitiveness/Responsiveness
- Incumbency Protection
- ...

Permissible Districting Plans

We want to partition a given geography (graph), at a given scale, into k pieces, satisfying some constraints:

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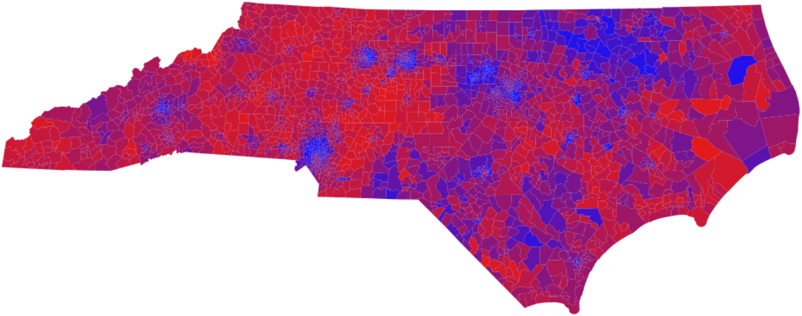
- Communities of Interest
- Municipal Boundaries
- Competitiveness/Responsiveness
- Incumbency Protection
- ...

Mathematical Formulation

Given a (connected) graph $G = (V, E)$:

- A k-**partition** $P = \{V_1, V_2, \ldots, V_k\}$ of G is a collection of disjoint subsets $V_i \subseteq V$ whose union is V.
- A partition P is **connected** if the subgraph induced by V_i is connected for all i.
- The **cut edges** of P are the edges (u, w) for which $u \in V_i$, $w \in V_j$, and $i \neq j$
- A partition P is ε-**balanced** if $\mu(1 - \varepsilon) \leq |V_i| \leq \mu(1 + \varepsilon)$ for all i where μ is the mean of the $|V_i|$'s
- An **equi–partition** is a 0-balanced partition

Vote Data



Data Availability

Example (What adjective best describes US Electoral data?)

Data Availability

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Abominable*

Data Availability

Example (What adjective best describes US Electoral data?)

Abominable*

* Alternatively, any adjective from "You're a mean one, Mr. Grinch." [1]

[1] Dr. Seuss, How the Grinch Stole Christmas, 1966.

Problem Setting

Input Data

- Fixed geography and level of resolution defines the dual graph
- Weights on the nodes determine voting and demographic data

Generate

- Graph partitions – Districting plans

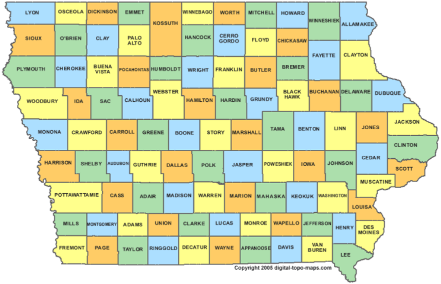
Analysis

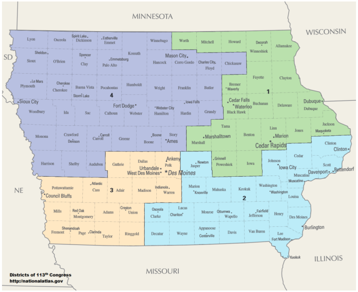
- Properties of the map
- (Expected) Partisan performance of the map

Aggregate

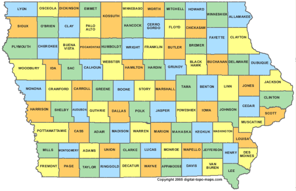
- Large ensembles
- Multiple statistics
- Baseline

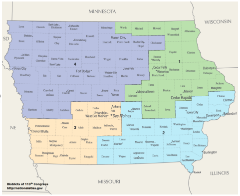
Example: Iowa





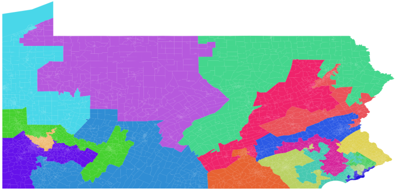
Example: Iowa

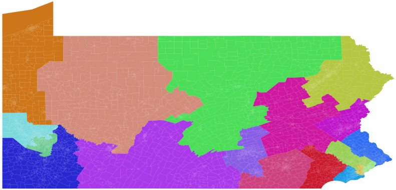




- 4 Congressional Districts, 100 House Districts, 50 Senate Districts
- House districts nest into Senate districts
- Congressional districts made out of counties
- Independent committee with legislative approval
- No partisan data allowed

Example: Pennsylvania



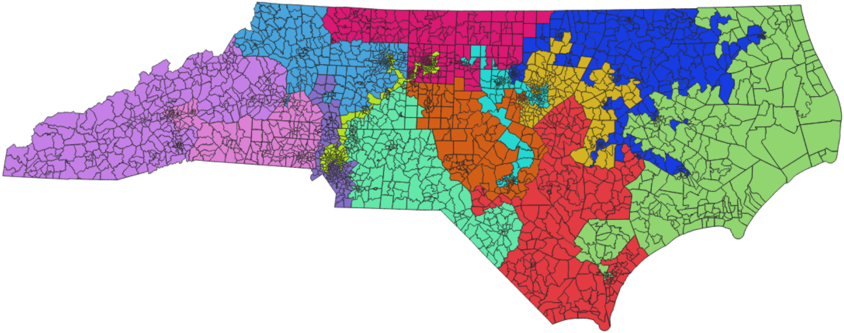


- 18 Congressional Districts, 203 House Districts, 50 Senate Districts
- Zero–balanced population
- Legislature draws congressional districts committee draws legislative districts
- Partisan behavior allowed

MORAL:

Computational Redistricting is NOT a solved problem!

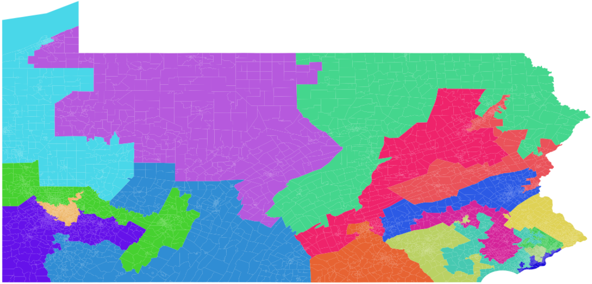
Ugly Shapes



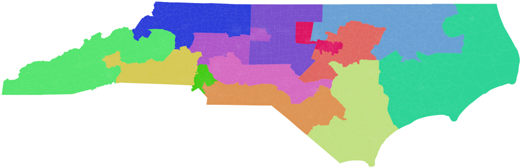
Ugly Shapes

(a) NC12 #1 (b) NC12 #2 (c) NC12 #12

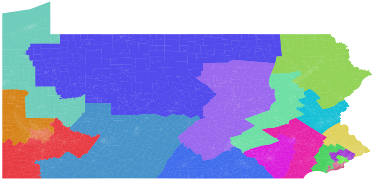
Ugly Shapes



Partisan Imbalance

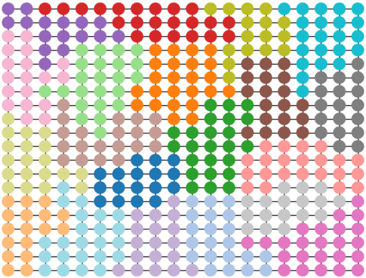


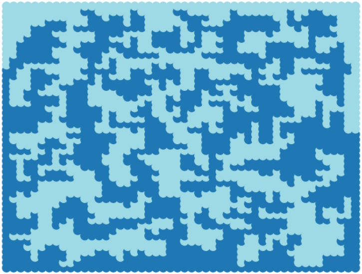
(a) NC16



(b) PA TS-Proposed

(Discrete) Total Perimeter





Polsby–Popper

Theorem (Isoperimetry)

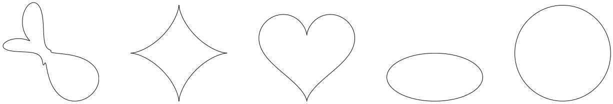
Let Ω be a bounded open subset of $\mathbb{R}^2$ with finite perimeter. Then:

$$4\pi A \leq P^2$$

Definition (Polsby–Popper)

The Polsby–Popper score of a district is:

$$PP(\Omega) = \frac{4\pi A}{P^2}$$

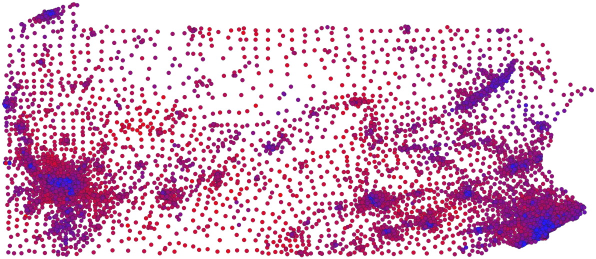


$\frac{4\pi A}{L^2} =$ 0.359 0.411 0.680 0.841 1.000

Total Variation Profile[1]

[1] D. DeFord, H. Lavenant, Z. Schutzman, and J. Solomon: Total Variation Isoperimetric Profiles, arXiv:1809.07943, 2018.

Partisanship Measures



Partisan Fairness

- MA
  - Duchin et al. (2018) Locating the representational baseline: Republicans in Massachusetts arXiv:1810.09051
  - Not all partisan outcomes are possible, given discretization
- MD
  - Two recent preprints claiming not gerrymandered
  - Court ruled one district unconstitutional
- NC/PA/WI
  - Heavy court involvement
  - Wide variance in partisan metrics

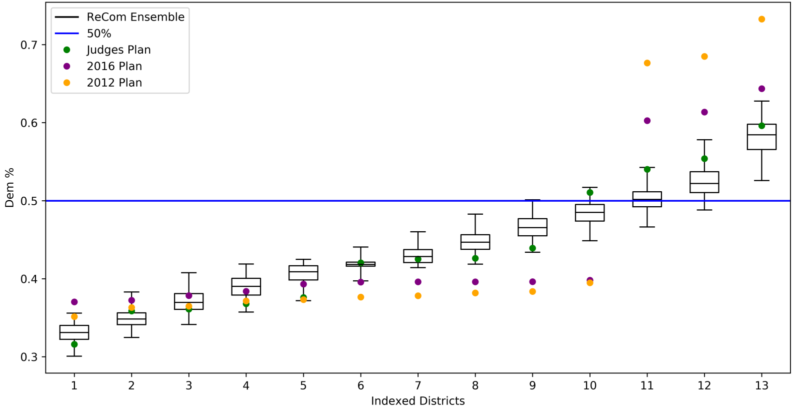
MORAL:

Computational Redistricting is **NOT** a solved problem!

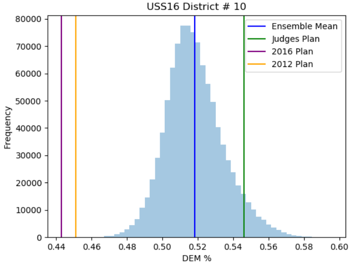
Ensemble Analysis

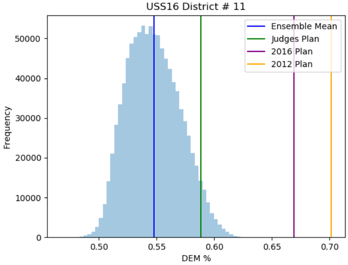
- The wide variety in rules applied to districting problems (even in the same state) means that any single measure of gerrymandering will be insufficient/exploitable
- Instead we want to do **outlier analysis** by comparing to large ensembles of other feasible plans.
- This allows us to understand the impacts of the underlying political and demographic geography on a wide collection of metrics.

Ensemble Example: NC

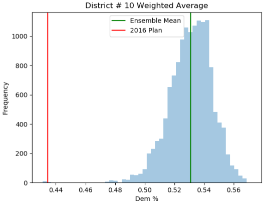


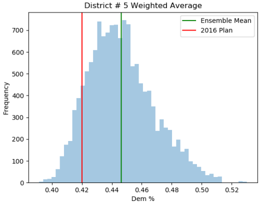
Ensemble Example: NC

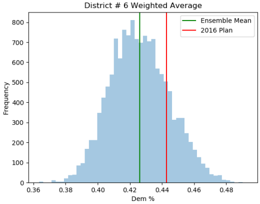


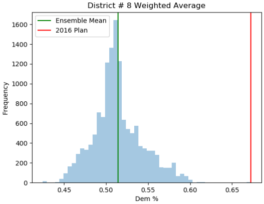


Ensemble Example: NC

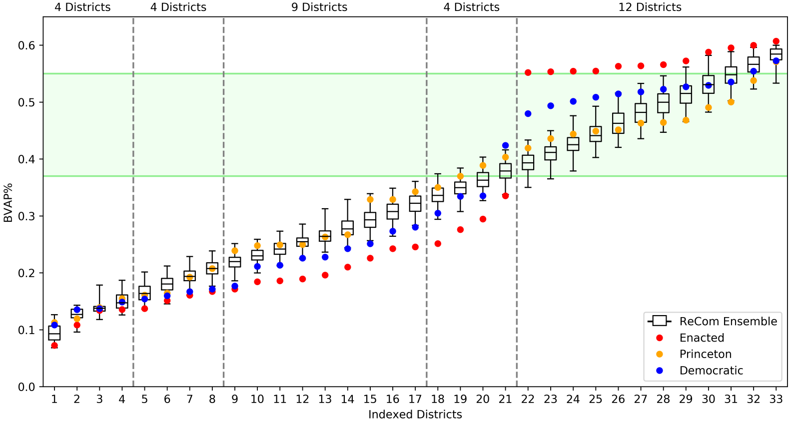




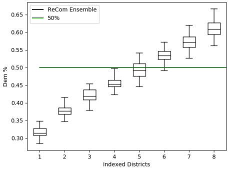




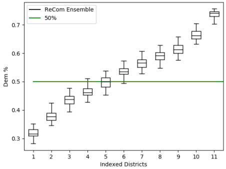
Ensemble Example: VA



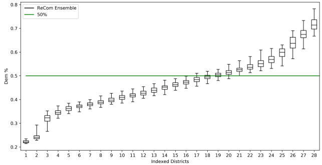
Ensemble Example: VA



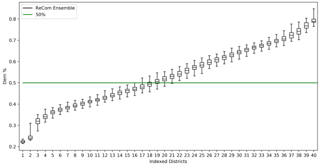
(a) No NoVa



(b) Full state

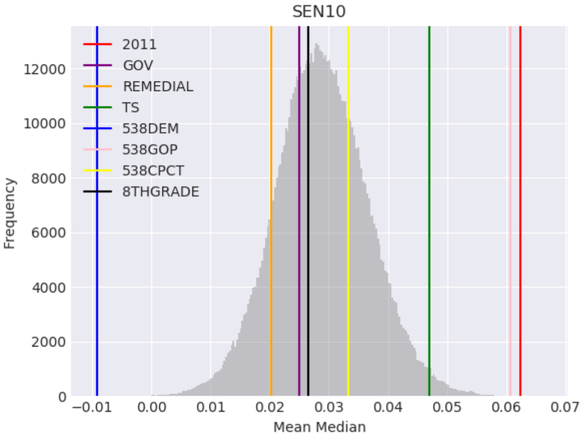


(c) No NoVa

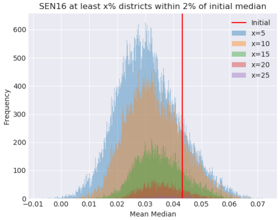


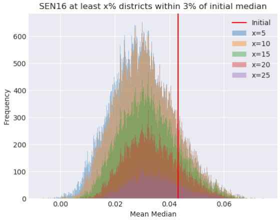
(d) Full state

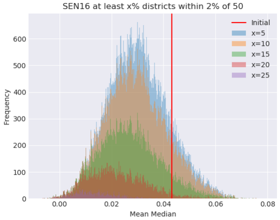
Ensemble Example: PA

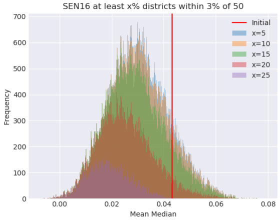


Ensemble Example: PA

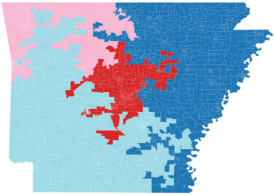


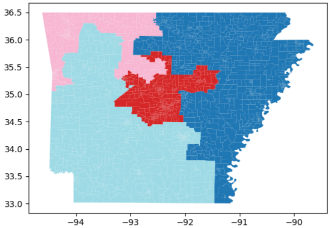






Which ensembles?





Ensembles in Practice

- The appeal of an ensemble method is that you get to control the input data very carefully
- However, just because a particular type of data was not considered doesn't mean that the outcome is necessarily "fair"
- There are lots of "random" methods for constructing districting plans
- Most don't offer any control over the distribution that you are drawing from

MCMC on partitions

1. Set constraints to define the state space
2. Start with an initial plan
3. Propose a modification
4. Verify that the modification satisfies the constraints
5. Accept using MH criterion
6. Repeat

MCMC on partitions

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Why?

MCMC on partitions

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Why?

- Control over sampling distribution and input data
- Possibility of local sampling
- Ergodic Theorem

MCMC on partitions

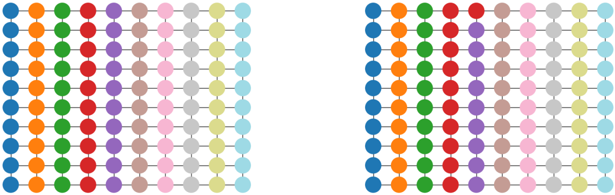
1. Set constraints to define the state space
2. Start with an initial plan
3. **Propose a modification**
4. Verify that the modification satisfies the constraints
5. Accept using MH criterion
6. Repeat

Why?

- Control over sampling distribution and input data
- Possibility of local sampling
- Ergodic Theorem

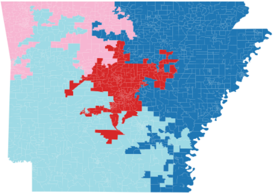
Single Edge Flip Proposals

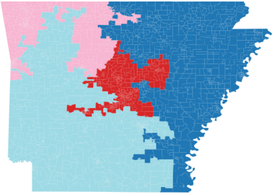
1. Uniformly choose a cut edge
2. Change one of the incident node assignments to the other



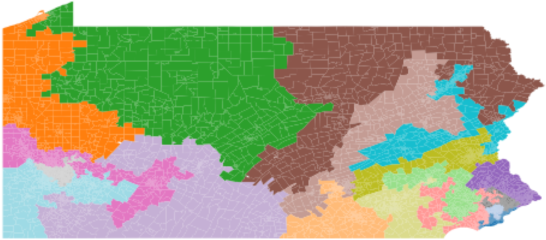
- Mattingly et al. (2017, 2018) Court cases in NC and WI.
- Pegden et al. Assessing significance in a Markov chain without mixing, PNAS, (2017). Court case in PA.

Single Edge Ensembles

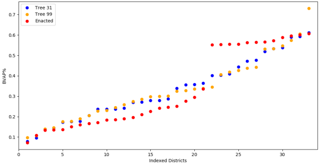




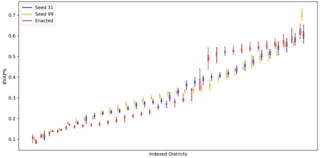
PA Single Edge Flip



Boundary Flip Mixing

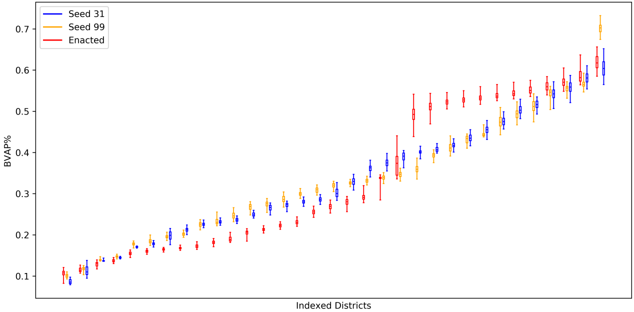


(a) Initial



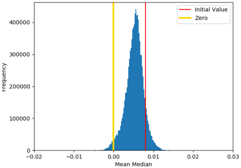
(b) 10,000,000 Flip Steps

Boundary Flip Mixing

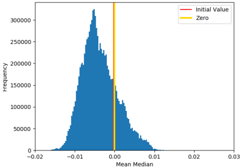


(b) 10,000,000 Flip Steps

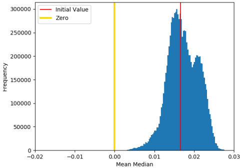
Boundary Flip Mean–Median



(a) Flip Seed31



(b) Flip Seed99



(c) Flip Enacted

Slowly Mixing Graph Families

Theorem (Najt 2019)

Let G be any connected graph. Then let $G^{(d)}$ be the graph obtained by replacing each edge by a doubled d-star. Then the flip walk on partitions of family of graphs $G^{(d)}_{d \geq 1}$ is slowly mixing, in the sense the Cheeger constant is decaying exponentially fast. More specifically:

$$H(Metagraph(G^{(d)}) = O(2^{-d})$$

Slow Mixing Example

Slow Mixing Example

Uniform Sampling of Contiguous Partitions

Theorem (Najt 2019)

Suppose that D is the class of connected planar graphs and $k \geq 2$. If there is a polynomial time algorithm to sample uniformly from any of:

- *the connected k-partitions of graphs in D,*
- *or the connected, 0-balanced k-partitions of graphs in D.*

then $RP = NP$.

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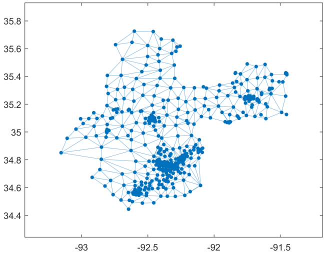
then $RP = NP$.

Remark

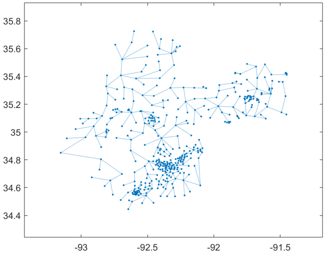
This theorem has various interesting extensions, including:

- *Connectivity constraints on D*
- *Degree bounds*
- *Subgraphs of lattices*
- *Distributions propotional to cut length*
- *Weaker population bounds*

Tree based methods

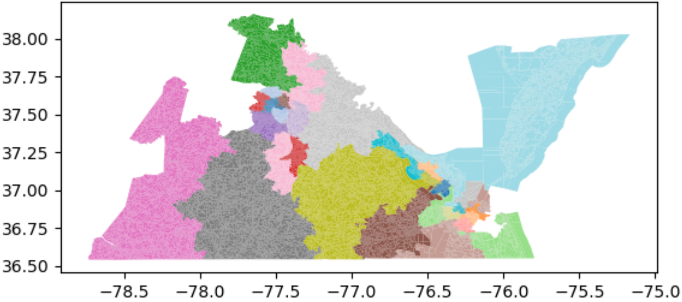


(a) District



(b) Spanning Tree

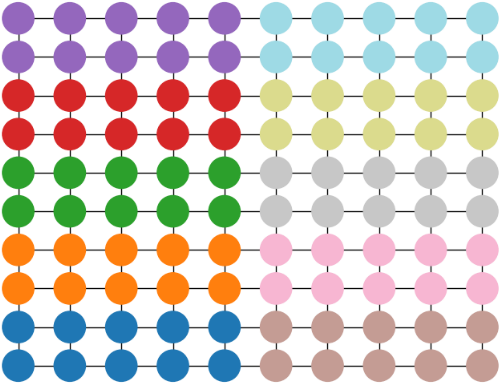
Tree Seeds Ensemble



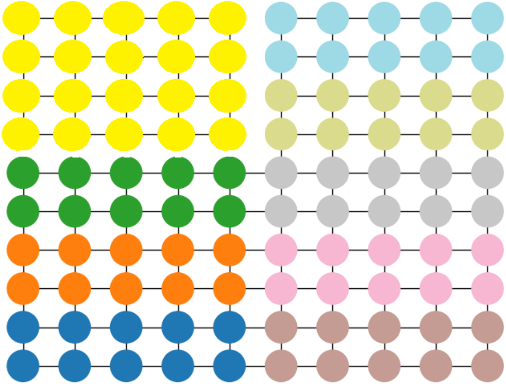
Recombination Steps

1. At each step, select two adjacent **districts**
2. Merge the subunits of those two districts
3. Draw a spanning tree for the new super–district
4. Delete an edge leaving two population balanced districts
5. Repeat
6. (Optional) Mix with single edge flips

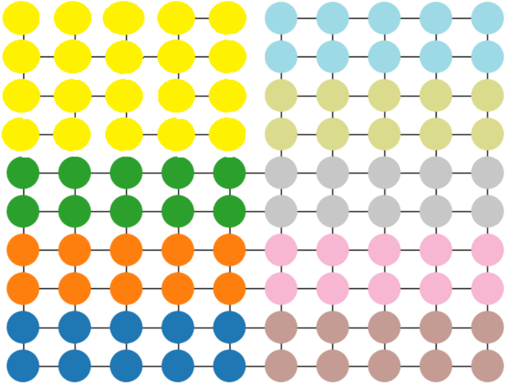
Recombination Step Example



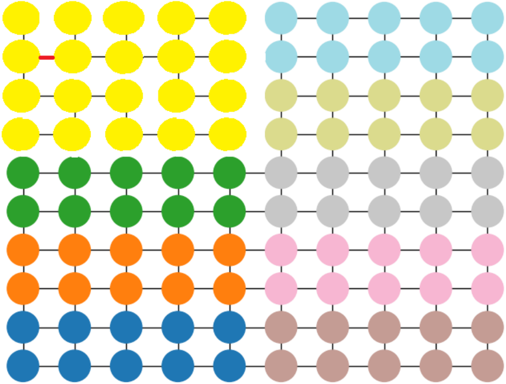
Recombination Step Example



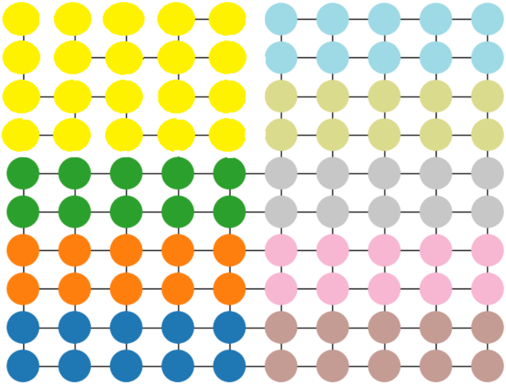
Recombination Step Example



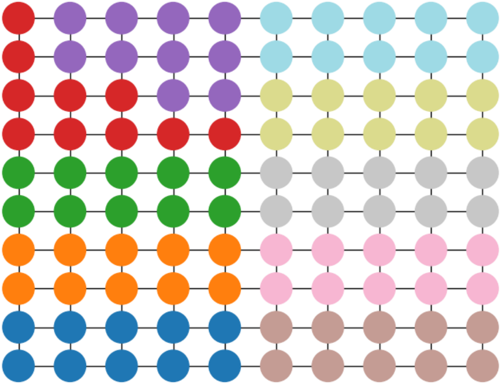
Recombination Step Example



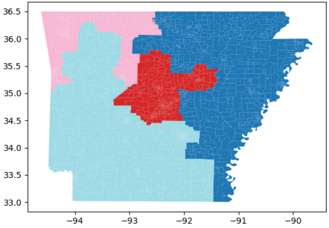
Recombination Step Example

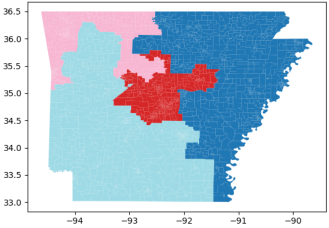


Recombination Step Example



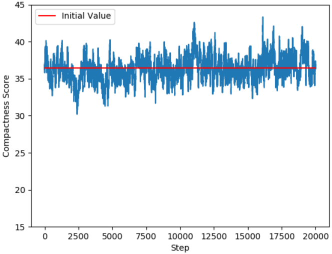
AR Ensembles



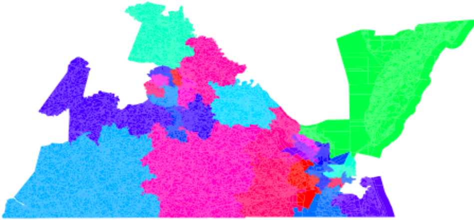


PA Recombination Steps

Recombination Distribution

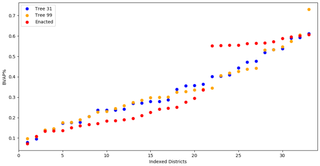


(a) Compactness

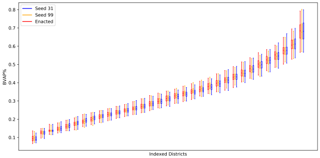


(b) 5702 cut edges

Recombination Mixing

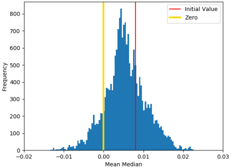


(a) Initial

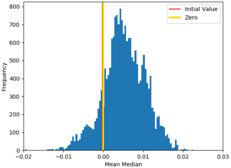


(b) 20,000 Recombination Steps

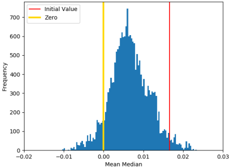
Recombination Mean–Median



(a) ReCom Seed31



(b) ReCom Seed99



(c) ReCom Enacted

General Tree Proposals

1. Form the induced subgraph on the complement of the cut edges
2. Add some subset of the cut edges
3. Uniformly select a maximal spanning forest
4. Apply a Markov chain on trees
5. Partition the spanning forest into k population balanced pieces

Special Cases

- Uniform Trees: Add all cut edges
- k–edges: Uniformly add k cut edges
- Recombination: Add all cut edges between one pair of districts.
- Super-Recombination: Take a maximal matching on the dual graph to the districts and add all cut edges between matched districts.
- Bounce Walk: Add a single cut edge between enough pairs of districts to make a tree in the dual graph of districts.

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Question

What are the steady state distributions (and mixing times) of these walks?

Tree Partitioning Questions

- Characterizing the distribution on partitions defined by cutting trees!
- How bad is the best cut?
- Criteria for determining when a tree is ε cuttable?
- Criteria for determining when all spanning trees of a graph are ε cuttable?
- How hard is it to find the mininum ε for which a cut exists?
- As a function of ε what proportion of spanning trees are cuttable?
- As a function of ε what proportion of edges in a given tree are cuttable?
- What is the fastest way to sample uniformly from $k-1$ balanced cut edges?

MORAL:

Computational Redistricting is NOT a solved problem!

The End

Thanks!

General Merge Proposals

1. At each step, select two adjacent **districts**
2. Merge the subunits of those two districts
3. Bipartition the new super–district
4. Repeat
5. (Optional) Mix with single edge flips

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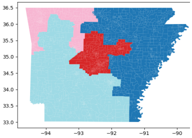
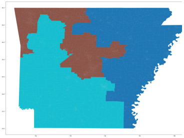
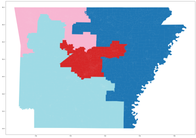


(a) Before



(b) During

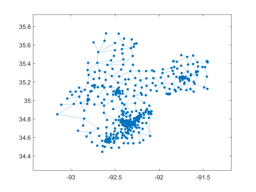


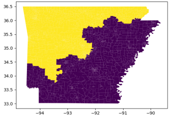
(c) After

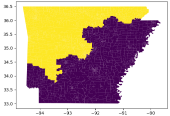
Tree Partitions

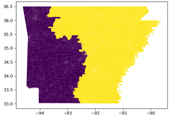
Method

- *Generate a uniform spanning tree*
- *Cut an edge that leaves population balanced components*









Flood Fill

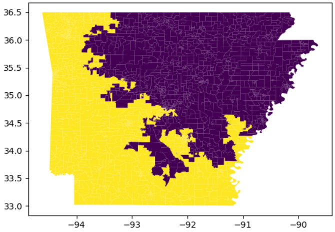
Method

- *Select a node at random*
- *Select a random neighbor of the current cluster*
- *Alternatively, generate a list of neighbors and append sequentially*
- *Add if population allows and doesn't disconnect the complement*
- *Repeat until population balanced*

Path Fill

Method

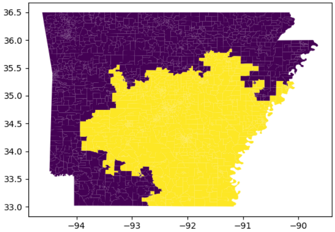
- *Start with an arbitrary node*
- *Select a node not in the district*
- *Add all the nodes on a shortest path from the new node to the district if it doesn't disconnect the complement or add too much to the population*
- *Repeat until population balanced*



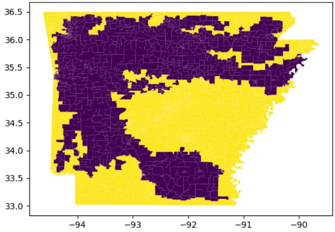
Agglomerative

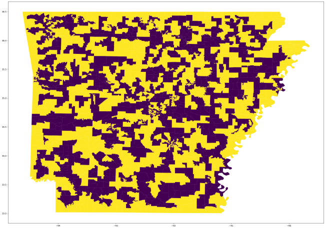
Method

- *Start with each node in own component*
- *Select an arbitrary edge between two components*
    - *Merge clusters if population allows and doesn't disconnect the complement*
    - *If population doesn't allow, delete edge*
    - *If merging would disconnect the graph, merge the smallest population component*
- *Repeat until only 2 clusters*



What can go wrong?





Min Cut

Method

- *Select random source and sink nodes*
- *Weight the edges in the graph by* $10^{min\ distance-3}$
- *Compute the min cut*
- *Repeat until population balanced*