Dynamics Modeling for Multiplex Networks

Daryl DeFord

Dartmouth College
Department of Mathematics

Joint Mathematics Meetings
AMS Contributed Paper Session on Networks and Data
San Diego, CA
January 10, 2018

Dartmouth

Abstract

Multiplex analyses of network data offer an important tool for leveraging the additional structure available in disaggregated data. In this talk, we will discuss the matched sum, a standard model for multiplex networks, describing the asymptotic properties of these graphs as the number of layers increases. These results allow us to characterize the pathological behavior that occurs when interlayer connections overwhelm the intralayer content. As an alternative to this structural approach, we present a family of dynamically motivated models that avoids these pathologies and provide related spectral results. Finally, we will apply these new methods to centrality and clustering problems for several data sets from the social sciences.

Outline

Dartmouth

What is a multiplex?

What is a multiplex?

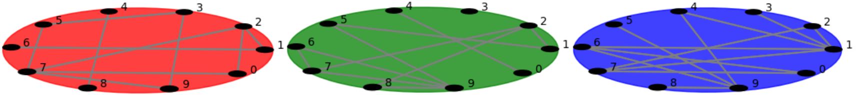
Definition

A *multiplex* is a collection of graphs all defined on the same node set.

What is a multiplex?

Definition

A *multiplex* is a collection of graphs all defined on the same node set.



(a) Family (b) Colleagues (c) Facebook

World Trade Web[1]

Figure: World trade networks

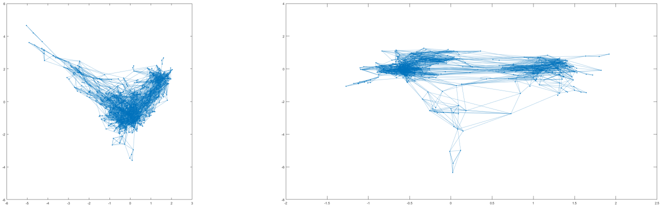
[1] R. Feenstra, R. Lipsey, H. Deng, A.C. Ma, and H. Mo. World Trade Flows: 1962-2000. NBER Working Paper 11040, (2005).

WTW Layers

Layer Description	Volume	% Total	Transitivity
Food and live animals	291554437	5.1	.82
Beverages and tobacco	48046852	0.9	.67
Crude materials	188946835	3.3	.79
Mineral fuels	565811660	10.0	.62
Animal and vegetable oils	14578671	0.3	.64
Chemicals	535703156	9.5	.83
Manufactured Goods	790582194	13.9	.87
Machinery	2387828874	42.1	.85
Miscellaneous manufacturing	736642890	13.0	.83
Other commodities	107685024	1.9	.56
Aggregate Trade	5667380593	100	.93

Table: Layer information for the 2000 World Trade Web.

Karnataka Village Data[1]



(a) Village 4 (b) Village 61

Figure: Karnataka Villages

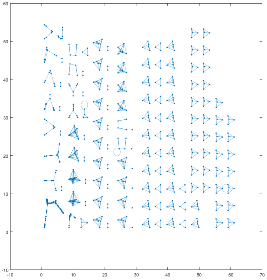
[1] A. Banerjee, A.G. Chandrasekhar, E. Duflo, and M.O. Jackson, The Diffusion of Microfinance. Science, (2013).

Village Layers

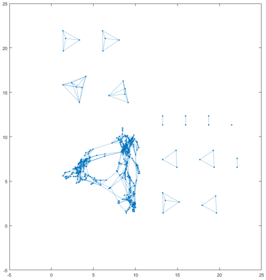
Layer	Village 4			Village 61		
Description	Density	Comp.	Giant %	Density	Comp.	Giant %
Borrow Money	.0082	26	.8354	.0108	15	.9188
Give Advice	.0077	49	.5892	.0098	34	.7377
Help Make Decisions	.0076	61	.1277	.0100	24	.8562
Borrow Kerosene or Rice	.0085	21	.8338	.0113	14	.9171
Lend Kerosene or Rice	.0086	22	.8308	.0113	14	.9255
Lend Money	.0081	14	.7908	.0107	17	.9036
Medical Advice	.0075	84	.2938	.0106	14	.9306
Friends	.0089	15	.9277	.0105	22	.8714
Relatives	.0085	29	.7231	.0105	26	.5448
Attend Temple With	.0073	117	.0462	.0089	108	.0372
Visit Their Home	.0087	15	.9185	.0116	11	.9475
Visit Your Home	.0088	16	.9108	.0117	11	.9492
Aggregate	.0121	3	.9862	.0155	8	.9679

Table: Layer information for two of the Karnataka Villages.

Medical Advice



(a) Village 4



(b) Village 61

Disjoint Layers

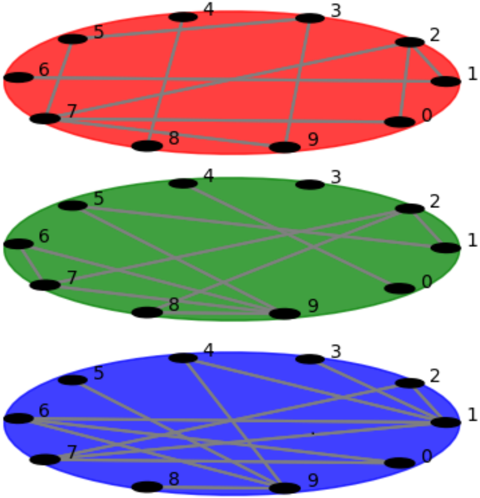
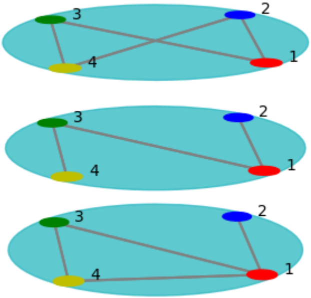
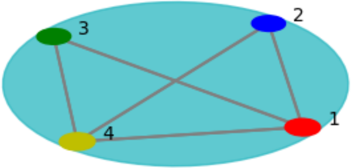


Figure: Disjoint Layers

Aggregate Models

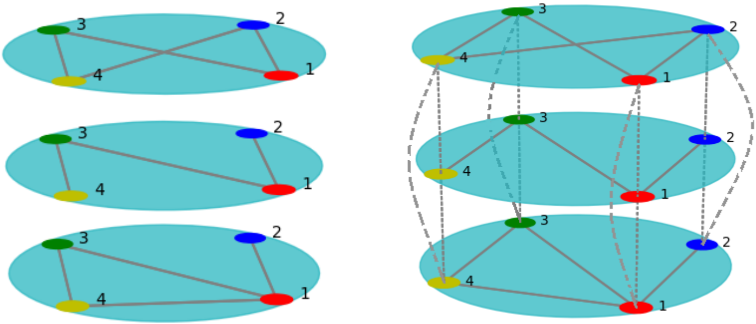


(a) Disjoint Layers



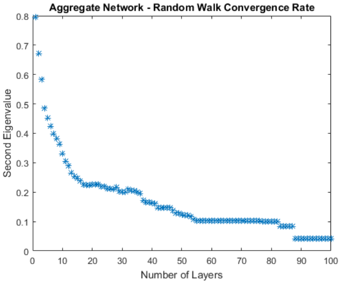
(b) Aggregate

Matched Sum

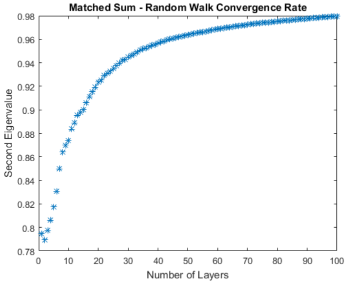


(a) Disjoint Layers (b) Matched Sum

Random Walk Convergence



(a) Aggregate



(b) Matched Sum

Dynamics on Multiplex Networks

Dynamics on Multiplex Networks

- Two types of interactions
  - Within the individual layers
  - Between the layers
- Effects should “pass through” nodes
- Two step iterative model

Dynamics on Multiplex Networks

- Two types of interactions
  - Within the individual layers
  - Between the layers
- Effects should "pass through" nodes
- Two step iterative model
- Symbolically:

$$v' = \mathscr{D}v$$

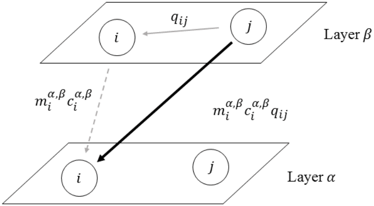
$$(v')_i^{\alpha} = \sum_{\beta=1}^{k} m_i^{\alpha,\beta} c_i^{\alpha,\beta} (Dv)_i^{\beta}$$

Dynamics on Multiplex Networks

- Two types of interactions
  - Within the individual layers
  - Between the layers
- Effects should "pass through" nodes
- Two step iterative model
- Symbolically:

$$v' = \mathscr{D}v$$

$$(v')_i^\alpha = \sum_{\beta=1}^{k} m_i^{\alpha,\beta} c_i^{\alpha,\beta} (Dv)_i^\beta$$



Matrix Realization

The matrix associated to the total operator also takes a convenient block form:

$$\begin{bmatrix} C_1D_1 & C_1D_2 & \cdots & C_1D_k \\ C_2D_1 & C_2D_2 & \cdots & C_2D_k \\ \vdots & \vdots & \vdots & \vdots \\ C_kD_1 & C_kD_2 & \cdots & C_kD_k \end{bmatrix}$$

Where the $\{D_i\}$ are the dynamical operators associated to the layers and the $\{C_i\}$ are the diagonal proportionality matrices.

Multiplex Random Walks

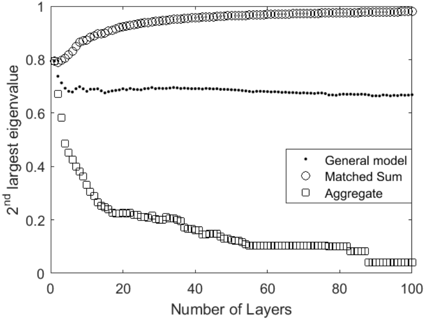


Figure: Comparison of random walk convergence for multiplex models.

Laplacian Eigenvalue Bounds

Theorem (D. 2016)

Let $\{\lambda_i\}$ *be the eigenvalues of* $\mathscr{D}$ *and* $\{\lambda_i^\alpha\}$ *be the eigenvalues of the* α*–layer Laplacian* D^α*. We have the following bounds:*

- *Fiedler Value:*

$$\max_\alpha(\lambda_F^\alpha) \leq k\lambda_F \leq \lambda_F^m + \sum_{\beta \neq m} \lambda_1^\beta$$

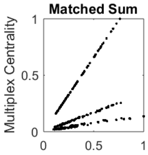
- *Leading Value:*

$$\max_i(\lambda_1^i) \leq k\lambda_1 \leq \sum_i \lambda_1^i$$

- *General Form:*

$$\max_i(\lambda_{n-j}^i) \leq k\lambda_{n-j} \leq \min_{J \vdash n+k-(j+1)} \left(\min_{\sigma \in S_n} \left(\sum_{\alpha=1}^{k} \lambda_{j_\alpha}^{\sigma(\alpha)} \right) \right)$$

Centrality Comparison



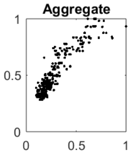


Figure: Comparison of multiplex eigenvector centrality scores. Varying the weighting scheme allows us to control how much mixing of centrality occurs between layers, while the matched sum model is just a linear transformation of the original rankings.

Centrality Comparison

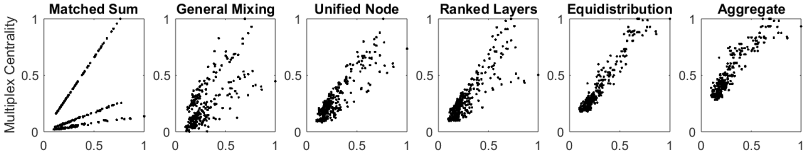
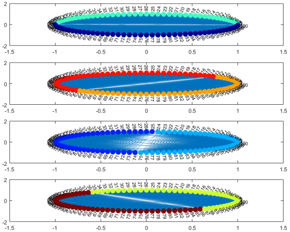
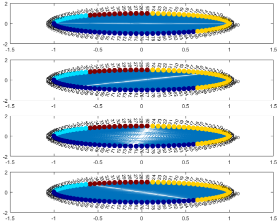


Figure: Comparison of multiplex eigenvector centrality scores. Varying the weighting scheme allows us to control how much mixing of centrality occurs between layers, while the matched sum model is just a linear transformation of the original rankings.

Clustering Comparison

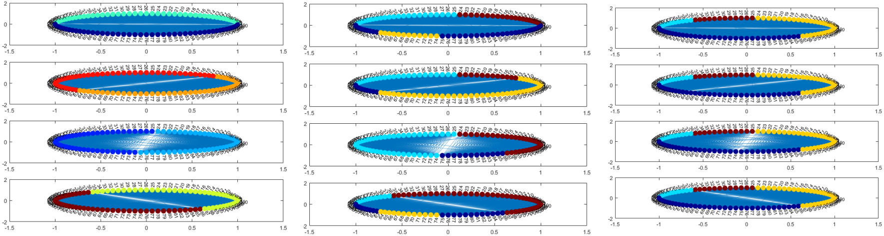


(a) Matched Sum



(c) Aggregate

Clustering Comparison



(a) Matched Sum (b) Dynamical Model (c) Aggregate

References

- M. Kivela, A. Arenas, M. Barthelemy, J. Gleeson, Y. Moreno, and M. Porter: *Multilayer Networks*, Journal of Complex Networks, 1–69, (2014).
- D. DeFord and S. Pauls: *A New Framework for Dynamical Models on Multiplex Networks*, Journal of Complex Networks, to appear, (2017), 29 pages.
- D. DeFord and S. Pauls: *Spectral Clustering Methods for Multiplex Networks*, submitted, arXiv:1703.05355, (2017), 22 pages.
- D. DeFord: *Multiplex Dynamics on the World Trade Web*, Proc. 6th International Conference on Complex Networks and Applications, Studies in Computational Intelligence, Springer, 1111–1123, (2018).

That's all...

Thank You!