Compactness Profiles and Reversible Sampling Methods for Plane and Graph Partitions

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MIT/Tufts
Metric Geometry and Gerrymandering Group

SAMSI Gerrymandering Workshop
Duke University
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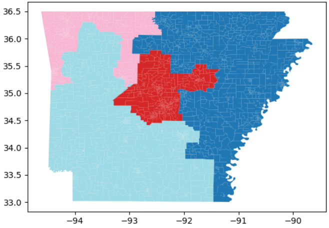
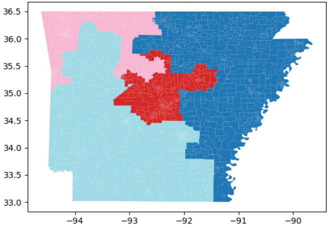
Outline

Collaborators

- Prof. Moon Duchin **Tufts Math**
- Prof. Justin Solomon **MIT CSAIL**
- Hugo Lavenant **Universitè Paris–Sud Math**
- Zachary Schutzman **UPenn CIS**
- Voting Rights Data Institute
    - 52 undergraduate and graduate students
    - 6–8 week summer program
    - `mggg.org`
    - `github.com/gerrymandr`

Arkansas


36.5
36.0
35.5
35.0
34.5
34.0
33.5
33.0
−94
−93
−92
−91
−90


36.5
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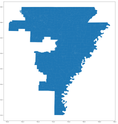
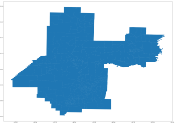
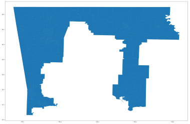
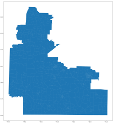
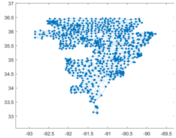
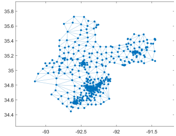
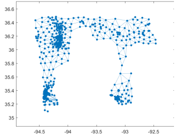
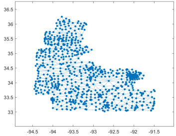
−90

Single Valued Compactness Measures

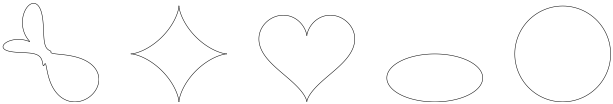












Isoperimetric Ratio



$$\frac{4\pi A}{L^2} = \quad 0.359 \quad 0.411 \quad 0.680 \quad 0.841 \quad 1.000$$

Figure: A variety of shapes marked with isoperimetric ratios $\frac{4\pi A}{L^2}$ marked.

Isoperimetric Profile

Definition (Isoperimetric Inequality)

Let $\Omega \subseteq \mathbb{R}^n$ to be a compact region whose boundary $\partial\Omega \subseteq \Omega$ is an $(n-1)$-dimensional hypersurface in $\mathbb{R}^n$

$$n \cdot \mathrm{vol}(\Omega)^{\frac{(n-1)}{n}} \cdot \mathrm{vol}(B(1, \mathbf{0}))^{\frac{1}{n}} \leq \mathrm{area}(\partial\Omega).$$

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Definition (Isoperimetric Profile)

With Ω as above and $t \in [0, \mathrm{vol}(\Omega)]$ we ask for the smallest surface area needed to enclose volume t completely within Ω:

$$I_\Omega(t) := \min\{\mathrm{area}(\partial\Sigma) : \Sigma \subseteq \Omega \text{ and } \mathrm{vol}(\Sigma) = t\}.$$

Total Variation

Definition (Three formulations of TV)

$$\mathrm{TV}[f] =$$

1. $$\sup\left\{\int_{\mathbb{R}^n}[f(x)\nabla\cdot\phi(x)]\,dx : \phi\in C_c^1(\mathbb{R}^n\to\mathbb{R}^n) \text{ and } \|\phi\|_\infty\le 1\right\}$$
2. $$\int_{\mathbb{R}^n}\|\nabla f\|_2\,dx$$
3. $$\int_0^{+\infty}\mathrm{area}(\partial\{f\ge s\})ds$$

Perimeter as Total Variation

Definition

For a region $\Sigma \subseteq \mathbb{R}^n$, denote its *indicator function* $\mathbb{1}_\Sigma$ via

$$\mathbb{1}_\Sigma(x) := \begin{cases} 1 & \text{if } x \in \Sigma \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Then, a consequence of the co-area formula is that

$$\text{area}(\partial\Sigma) = \text{TV}[\mathbb{1}_\Sigma]. \tag{2}$$

TV Relaxation

Definition (Isoperimetric Profile)

$$I_\Omega(t) = \begin{cases} \inf_{f \in L^1(\mathbb{R}^n)} & \mathrm{TV}[f] \\ \text{subject to} & \int_{\mathbb{R}^n} f(x)\,dx = t \\ & 0 \leq f \leq \mathbb{1}_\Omega \\ & f(x) \in \{0,1\}\ \forall x \in \mathbb{R}^n. \end{cases}$$

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Definition (TV Profile)

$$I_\Omega^{\mathrm{TV}}(t) := \begin{cases} \min_{f \in L^1(\mathbb{R}^n)} & \mathrm{TV}[f] \\ \text{subject to} & \int_{\mathbb{R}^n} f(x)\,dx = t \\ & 0 \leq f \leq \mathbb{1}_\Omega. \end{cases}$$

Example: Circle

Proposition

For all (Ω, t), *we have* $I_\Omega^{\mathrm{TV}}(t) \leq I_\Omega(t)$.

Example (Circle)

Suppose $\Omega \subset \mathbb{R}^2$ *is a circle of radius* R, *and take* $t = \pi r^2$ *for* $r \in (0, R)$. *In this case, by the isoperimetric inequality we know* $I_\Omega(t) = 2\pi r$. *But suppose we take* $f(x) \equiv \dfrac{r^2}{R^2}$. *By the co-area formula*

$$I_\Omega^{\mathrm{TV}}(t) \leq \mathrm{TV}[f] = 2\pi R \cdot \frac{r^2}{R^2} = 2\pi r \cdot \frac{r}{R} < I_\Omega(t).$$

Hence, our relaxation is not tight.

Isoperimetry and Convexity

Proposition (Isoperimetry)

Suppose $B \subset \mathbb{R}^n$ is a ball whose volume matches $\mathrm{vol}(\Omega)$. Then, for all $t \in [0, \mathrm{vol}(\Omega)]$, we have $I_B^{\mathrm{TV}}(t) \leq I_\Omega^{\mathrm{TV}}(t)$, and if the equality holds for some $t > 0$ then Ω is a ball.

Proposition (Convexity)

$I_\Omega^{\mathrm{TV}}(t)$ is a convex function of t.

Proposition (Convex Envelope)

The function I_Ω^{TV} is the lower convex envelope of I_Ω.

Minimizer Structure

Proposition (Distinguished Solutions)

There exists a family $(f_t)_{t\in[0,1]}$ such that:

- *For any $t \in [0,1]$, the function $f_t \in L^1(\mathbb{R}^n)$ satisfies $0 \leq f_t \leq \mathbb{1}_\Omega$, $\int_{\mathbb{R}^n} f_t(x)\,dx = t$ and $\mathrm{TV}(f_t) = I_\Omega^{\mathrm{TV}}(t)$.*
- *For any $t \in [0,1]$, there exist $v_t \in (0,1)$ such that f_t takes its values in $\{0, v_t, 1\}$.*
- *For a.e. $x \in \Omega$, the function $t \to f_t(x)$ is increasing.*

Simple Animation

Cheeger Sets

Definition (Cheeger Constant)

The Cheeger constant of Ω, denoted by $h_1(\Omega)$, is defined as

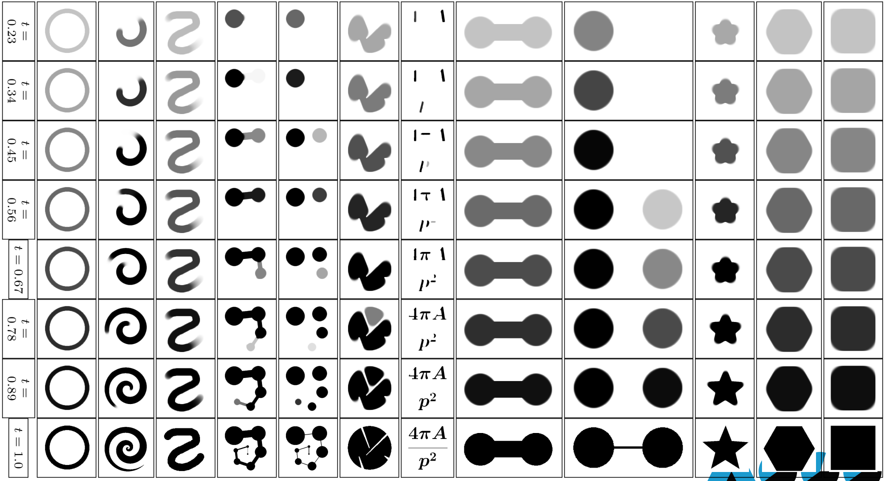
$$h_1(\Omega) := \inf_{\tilde{\Sigma} \subseteq \Omega} \frac{\operatorname{area}(\partial\tilde{\Sigma})}{\operatorname{vol}(\tilde{\Sigma})},$$

and a subset $\Sigma \subseteq \Omega$ such that $h_1(\Omega) = \frac{\operatorname{area}(\partial\Sigma)}{\operatorname{vol}(\Sigma)}$ is known as a Cheeger set of Ω.

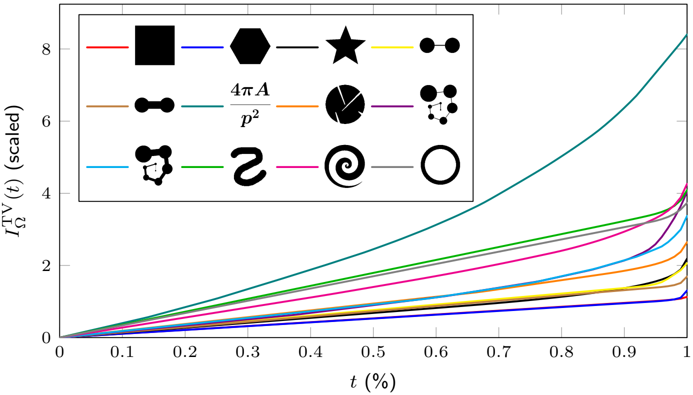
Proposition (Small t)

Let Ω be compact, let $h_1(\Omega)$ be the Cheeger constant of Ω, and let Σ be a Cheeger set of Ω. Then for any $t \leq \operatorname{vol}(\Sigma)$, we have $I_\Omega^{\mathrm{TV}}(t) = h_1(\Omega)t$, and a solution f is given by $f := \frac{t}{\operatorname{vol}(\Sigma)} \cdot \mathbb{1}_\Sigma$.

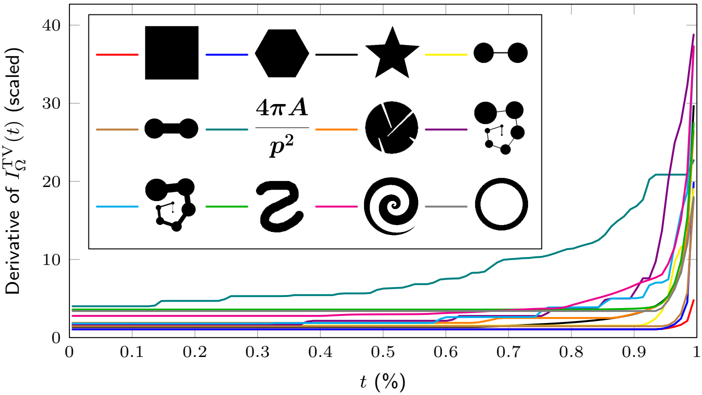
Synthetic Examples



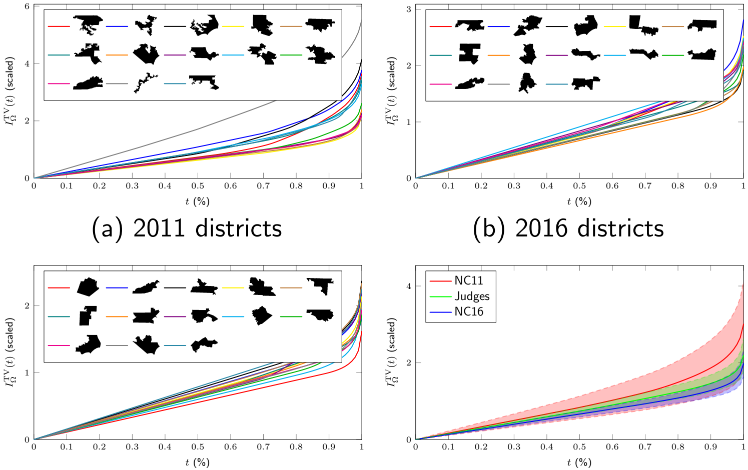
Synthetic Profiles



Synthetic Derivatives



North Carolina



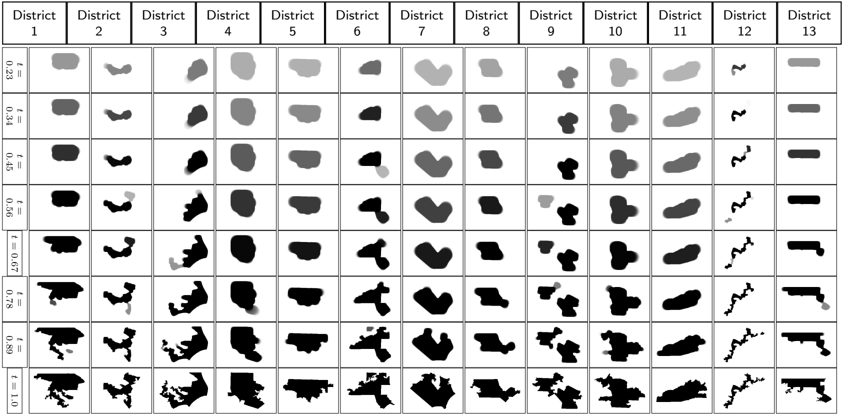
(a) 2011 districts

(b) 2016 districts

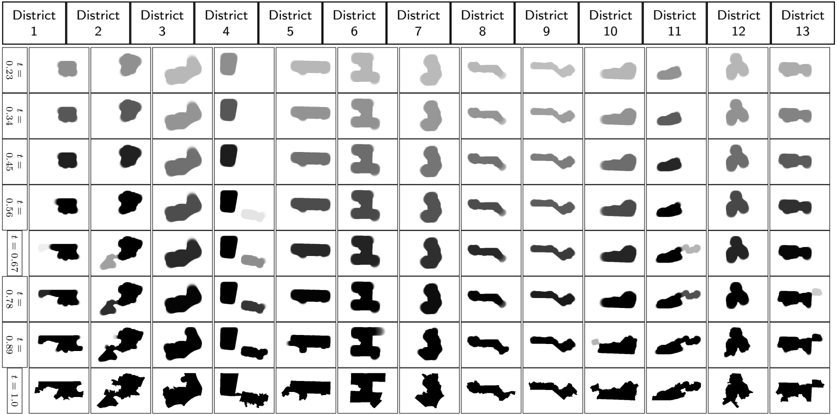
(c) Judges plan

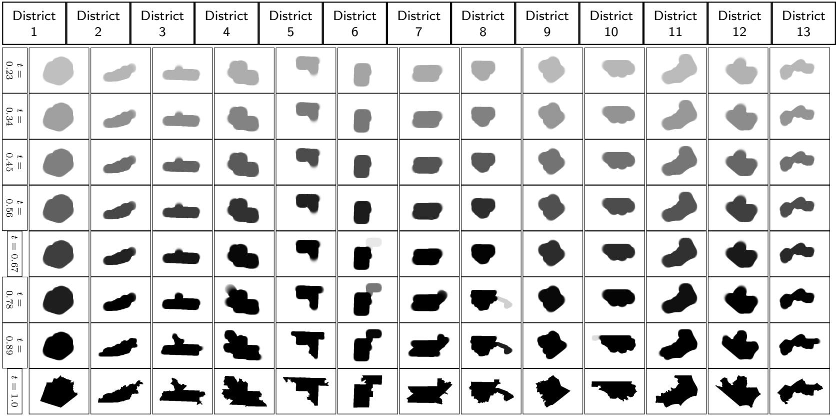
(d) Comparison

NC 2011 Districts

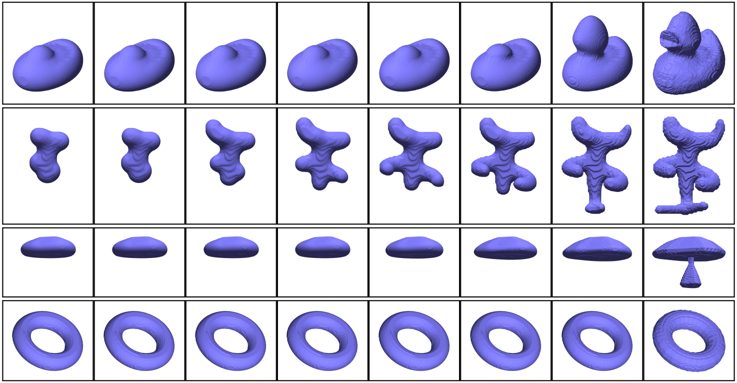


NC 2016 Districts




District 1
District 2
District 3
District 4
District 5
District 6
District 7
District 8
District 9
District 10
District 11
District 12
District 13
$t = 0.23$
$t = 0.34$
$t = 0.45$
$t = 0.56$
$t = 0.67$
$t = 0.78$
$t = 0.89$
$t = 1.0$

Higher Dimensions



$t = 0.23$	$t = 0.34$	$t = 0.45$	$t = 0.56$	$t = 0.67$	$t = 0.78$	$t = 0.89$	$t = 1.0$

Other Formulations

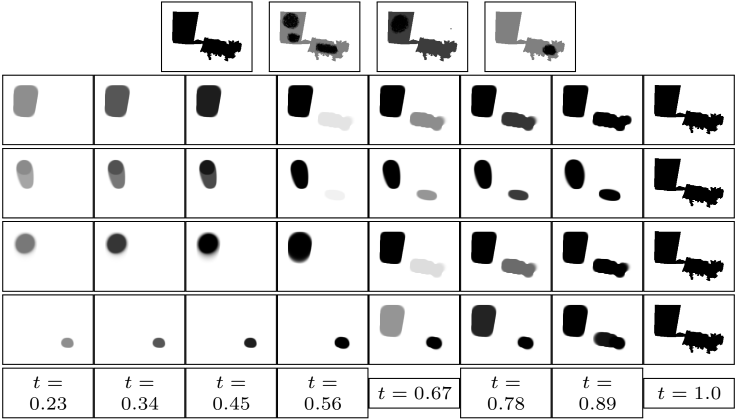
Definition (Population Measure)

$$I^{\mathrm{TV}}_{\Omega,\rho}(t) := \begin{cases} \min_{f\in L^1(\mathbb{R}^n)} & \mathrm{TV}[f] \\ \text{subject to} & \int_{\mathbb{R}^n} f(x)\, d\rho(x) = t \\ & 0 \le f \le \mathbb{1}_\Omega . \end{cases}$$

Definition (Discrete)

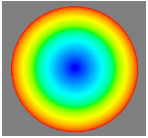
$$I^{\mathrm{TV}}_{V_0}(t) := \begin{cases} \min_{f\in\mathbb{R}^V} & \sum_{(v,w)\in E} |f(v) - f(w)| \\ \text{subject to} & \sum_{v\in V_0} f(v) = t|V_0| \\ & f(v) = 0 \ \forall v \notin V_0 \\ & f(v) \in [0,1] \ \forall v \in V. \end{cases}$$

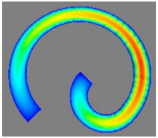
Synthetic Cities

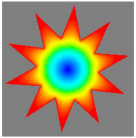


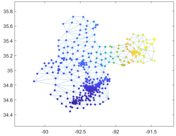
Discrete Animation

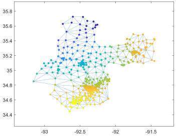
Heat Trace Kernel (VRDI)

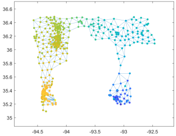


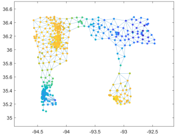




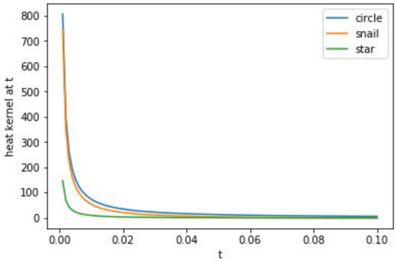




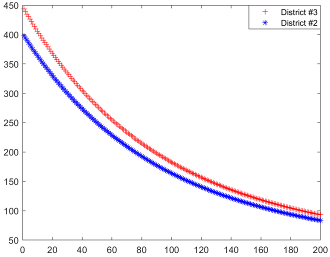




Heat Trace Kernel (VRDI)



(a) Continuous

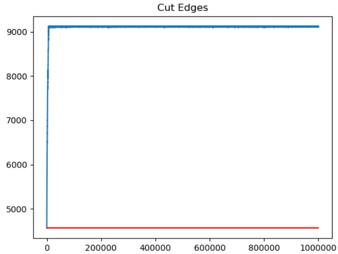


(b) Discrete

Mean Curvature Flow (VRDI)

Mean Curvature Flow (VRDI)

Single Edge Flip



Merge steps

1. At each step, select two adjacent **districts**
2. Merge the subunits of those two districts
3. Bipartition the new super–district
4. Repeat
5. (Optional) Mix with single edge flips

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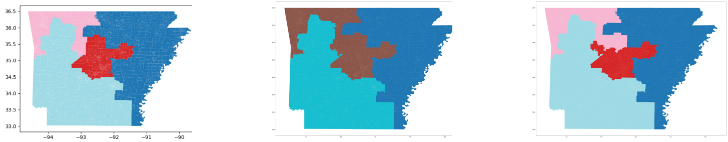
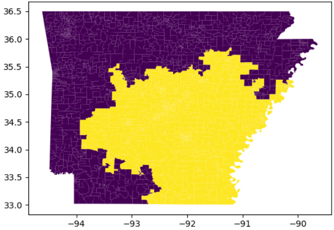


(a) Before (b) During (c) After

Agglomerative

Method

- *Start with each node in own component*
- *Select an arbitrary edge between two components*
  - *Merge clusters if population allows and doesn't disconnect the complement*
  - *If population doesn't allow, delete edge*
  - *If merging would disconnect the graph, merge the smallest population component*
- *Repeat until only 2 clusters*



Flood Fill

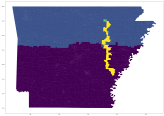
Method

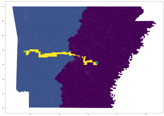
- *Select a node at random*
- *Select a random neighbor of the current cluster*
- *Add if population allows and doesn't disconnect the complement*
- *Repeat until population balanced*

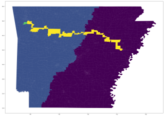
Min Cut

Method

- *Select random source and sink nodes*
- *Weight the edges in the graph by* $10^{min\ distance-3}$
- *Compute the min cut*
- *Repeat until population balanced*



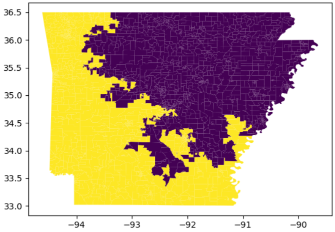




Path Fill

Method

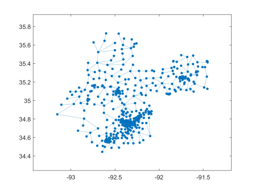
- *Start with an arbitrary node*
- *Select a node not in the district*
- *Add all the nodes on a shortest path from the new node to the district if it doesn't disconnect the complement or add too much to the population*
- *Repeat until population balanced*

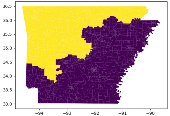


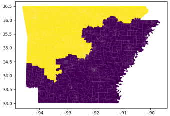
Tree Partitions

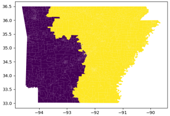
Method

- *Generate a uniform spanning tree*
- *Cut an edge that leaves population balanced components*









Pennsylvania

The end!

Thanks!