Eugene Wigner [4]:

The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning.[1]

[1] E. Wigner: *The unreasonable effectiveness of mathematics in the natural sciences*, Communications in Pure and Applied Mathematics, XIII, (1960), 1–14.

Total Dynamics on Multiplex Networks

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Graduate Student Combinatorics Conference
University of Kentucky
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Abstract

Analyzing combinatorially motivated dynamics on graphs leads to some of the most important tools and invariants in complex networks. In this talk we present an algebraic method for extending these techniques to multiplex networks, in terms of an operator that connects the endogenous and exogenous dynamics on the graph. We will provide a thorough analysis of the derived operators, including eigenvalue bounds, for the most commonly studied families of dynamics, such as random walks and diffusion via the graph Laplacian, and show how this method can be applied in economics, transportation, and social networks.

Outline

Dartmouth

Key Idea

Spectral graph theory studies invariants of graphs using the spectral structure of associated matrices.

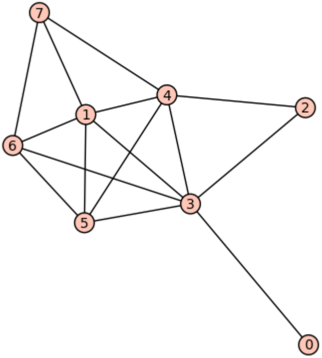
Process:

- Structural Representation
- Dynamical Interpretation
- Spectral Analysis

Tools

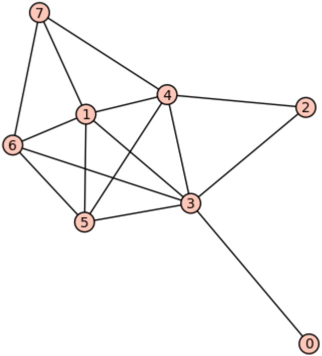
- Linear Algebra
  - Orthogonal Diagonalization
  - Perron–Frobenius
  - Spectral Analysis
- Analysis
  - Matrix Calculus
  - Perturbation Theory
- Riemannian Geometry
  - Cheeger Inequalities
  - Flows

Degree Matrix



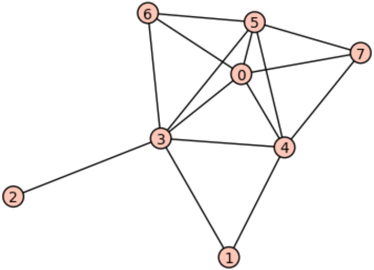
$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

Adjacency Matrix



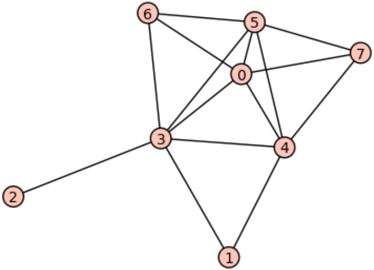
$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Incidence Matrix



$$N = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Laplacian



$$L = \begin{bmatrix} 5 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 2 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 6 & -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & -1 & 5 & -1 & 0 & -1 \\ -1 & 0 & 0 & -1 & -1 & 5 & -1 & -1 \\ -1 & 0 & 0 & -1 & 0 & -1 & 3 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & 3 \end{bmatrix}$$

Dynamics on Networks

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- Adjacency Matrix
  - $v_i = \sum_{i \sim j} v_j = \sum_j A_{i,j} v_j$
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  - Normalized (AD^{-1}) leads to random walks

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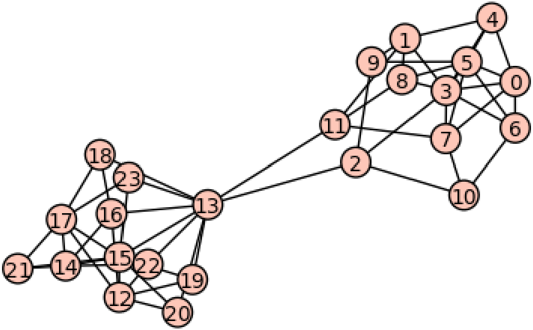
- Adjacency Matrix
  - $v_i = \sum_{i \sim j} v_j = \sum_j A_{i,j} v_j$
  - Flows across edges
  - Normalized (AD^{-1}) leads to random walks
- Laplacian
  - Heat flow
  - Isoperimetric clustering
  - Random walks $AD^{-1} = D^{-\frac{1}{2}}(I - D^{-\frac{1}{2}} L D^{-\frac{1}{2}}) D^{\frac{1}{2}}$

Clustering



Fan Chung [1]

Roughly speaking, half of the main problems of spectral theory lie in deriving bounds on the distributions of eigenvalues. The other half concern the impact and consequences of the eigenvalue bounds as well as their applications.[2]

[2] F. Chung: *Spectral Graph Theory*, AMS, (1997).

Spectral Results[3][4]

[3] F. Chung: *Spectral Graph Theory*, AMS, (1997)

[4] R. Brualdi: *The Mutually Beneficial Relationship of Graphs and Matrices*, AMS, (2011).

Spectral Results[3][4]

- Adjacency Matrix
  - Number of closed walks of length k is $\mathrm{Trace}(A^k)$
  - $\frac{1}{n}(\sum_{i=1}^n d_i) \leq \lambda_1(A) \leq \max_i(d_i)$
  - $|\{\lambda_i(A)\}| \geq \mathrm{diam}(G)$
  - $\alpha(G) \leq n - \max(|\{\lambda_i(A) > 0\}|, |\{\lambda_i(A) < 0\}|$
  - ...

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  - ...
- Laplacian
  - The multiplicity of 0 as an eigenvalue of L is the number of connected components of G
  - Algebraic connectivity $\mu(G) = \lambda_{n-1}(L)$
  - $\mu(G) \leq \nu(G) \leq \varepsilon(G)$
  - $\mu(G) \geq \frac{1}{\mathrm{diam}(G)\,\mathrm{vol}(G)}$
  - ...

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Multiplex Definition

Definition

A multiplex is a collection of graphs all defined on the same node set.

The motivations for studying these objects are mostly practical:

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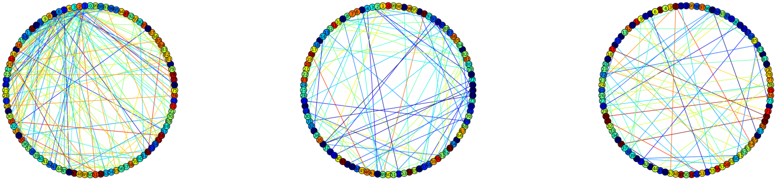
The motivations for studying these objects are mostly practical:

- Trade networks
- Social networks
- Neural networks
- Anonymity networks

Multiplex Example



(a) Layer 1 (b) Layer 2 (c) Layer 3

Figure : Three layers of a World Trade Web model

Motivation

Many of these structures have intrinsic dynamics that distinguish between connections between distinct nodes and connections between copies of the same node. Early approaches to studying graph problems in this context tried to address this problem from a structural perspective[5] (summing matrices or adding edges between copies). These approaches tend to distort the metrics of interest by conflating the intra and inter relationships.

[5] S. Gomez, A. Diaz-Guilera, J. Gomez-Gardenes, C.J. Perez-Vicente, Y. Moreno, and A. Arenas: *Diffusion Dynamics on Multiplex Networks*, Physical Review Letters, 110, (2013).

Algebraic Approach

Instead of trying to add new structural components we connect the dynamics using a collection of scaled orthogonal projections. To each node, we associate a projection operator P_n that gathers the information stored at each node and proportionally redistributes it among the copies. This allows us to respect the independence of the endogenous dynamics.

Linear Case

In this linear case this is particularly convenient. Given a collection of operators D_i on our layers, this is equivalent to constructing the new operator:

$$M = \begin{bmatrix} \alpha_{1,1}C_1D_1 & \alpha_{1,2}C_1D_2 & \cdots & \alpha_{1,k}C_1D_k \\ \alpha_{2,1}C_2D_1 & \alpha_{2,2}C_2D_2 & \cdots & \alpha_{2,k}C_2D_k \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{k,1}C_kD_1 & \alpha_{k,2}C_kD_2 & \cdots & \alpha_{k,k}C_kD_k \end{bmatrix}$$

where the $C_i = \mathrm{diag}(c_{i,1}, c_{i,2} \ldots, c_{i,\ell})$ represent the coefficients for the node projections with the condition that $\sum_j c_{i,j} = 1$ for all i.

Interpretation

This construction gives us an operator to study, instead of a structural representation, that captures both types of dynamics in a natural way. The next step is to relate the spectrum of this operator to the graph invariants.

Preserved Properties

The types of questions we are interested in depend on the initial properties of the dynamics, like positive definiteness or stochasticity. In order to interpret the results about our operator it must share these properties.

Theorem (Condensed)

If the original dynamics are
{Irreducible, Primitive, Stochastic, Positive(negative) (semi–)Definite}
then M is
{Irreducible, Primitive, Stochastic, Positive(negative) (semi–)Definite}.

Eigenvalue Relations

Theorem (Simplest Case)

If $c_{i,j} = \frac{1}{k}$ for all i, j then the non–zero eigenvalues of M are the eigenvalues of $a\sum_{\ell=1}^{k} D_\ell$.

Theorem (General Case)

If $c_{i,j} \neq 0$ for all i, j then the non–zero eigenvalues of M are the eigenvalues of $C_1(\sum_{\ell=1}^{k} D_\ell C_\ell)C_1^{-1}$.

Laplacian Eigenvalue Bounds

In the case where the individual layer dynamics are the respective Laplacians, more can be said about the eigenvalues of interest:

- Fiedler Value: $\max_i(\lambda_f^i) \leq \lambda_f \leq \min_i(\lambda_1^i) + \sum_{j \neq \ell} \lambda_f^j$
- Leading Value: $\min_i(\lambda_1^i) \leq \lambda_1 \leq \sum_i \lambda_1^i$
- Synchronization: Directly computed as the quotient of the previous two bounds

References

F. Chung: *Spectral Graph Theory*, AMS, (1997).

R. Brualdi: *The Mutually Beneficial Relationship of Graphs and Matrices*, AMS, (2011).

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E. Wigner: *The Unreasonable effectiveness of mathematics in the natural sciences*, Communivations in Pure and Applied Mathematics, XIII, (1960), 1–14.

That's all...

Thank You!