Èdouard Lucas:

The theory of recurrent sequences is an inexhaustible mine which contains all the properties of numbers; by calculating the successive terms of such sequences, decomposing them into their prime factors and seeking out by experimentation the laws of appearance and reproduction of the prime numbers, one can advance in a systematic manner the study of the properties of numbers and their application to all branches of mathematics.

Counting Combinatorial Rearrangements: What are the right questions?

Daryl DeFord

Dartmouth College

September 18, 2013

Dartmouth

Abstract

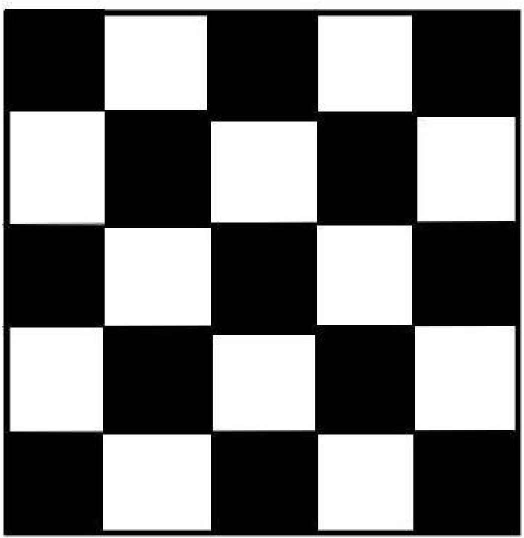
In this talk I will introduce a simple combinatorial model based on seating rearrangements in a classroom. The model has a natural extension to arbitrary graphs and connects with various traditional tiling problems. Many of the solutions to these problems can be expressed in terms of simple and well-known recurrent sequences like the Fibonacci numbers and the Pell numbers. In addition to presenting simple problems and solutions, I will also prove several general theorems about this model and discuss some interesting consequences of attempting to "fix" one of these theorems.

Outline

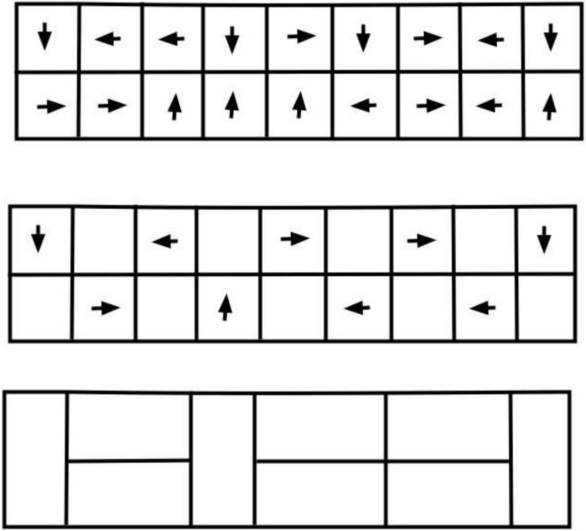
Original Problem (Honsberger)

A classroom has 5 rows of 5 desks per row. The teacher requests each pupil to change his seat by going either to the seat in front, the one behind, the one to his left, or the one to his right (of course not all these options are possible to all students). Determine whether or not his directive can be carried out.

Original Problem



Seating Rearrangements and Tilings



Arbitrary Graphs

In order to count rearrangements on arbitrary graphs, we constructed the following problem statement:

Problem

Given a graph, place a marker on each vertex. We want to count the number of legitimate "rearrangements" of these markers subject to the following rules:

- *Each marker must move to an adjacent vertex.*
- *After all of the markers have moved, each vertex must contain exactly one marker.*

To permit markers to **either** *remain on their vertex or move to an adjacent vertex, add a self-loop to each vertex (forming a pseudograph)*

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Digraphs

With this problem statement we can describe these rearrangements mathematically as follows:

- Given a graph G, construct $\overleftrightarrow{G}$, by replacing each edge in G with a two directed edges (one in each orientation).
- Then, each rearrangement on G corresponds to a cycle cover of $\overleftrightarrow{G}$.

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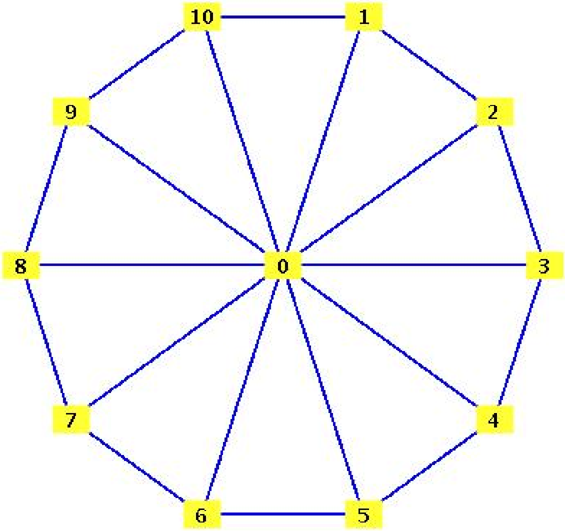

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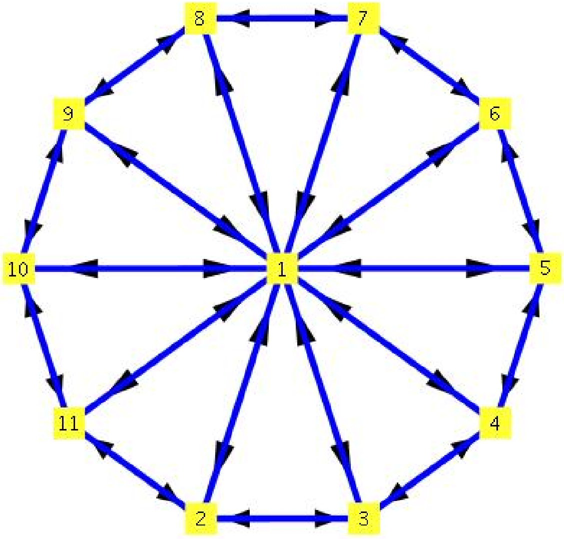


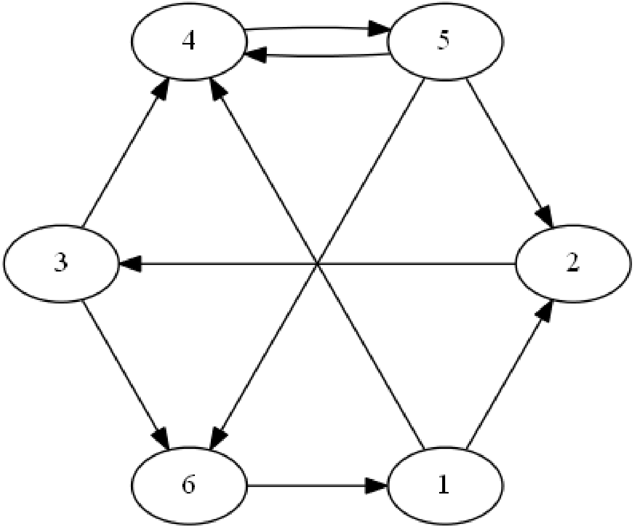
Cycle Covers

Definition

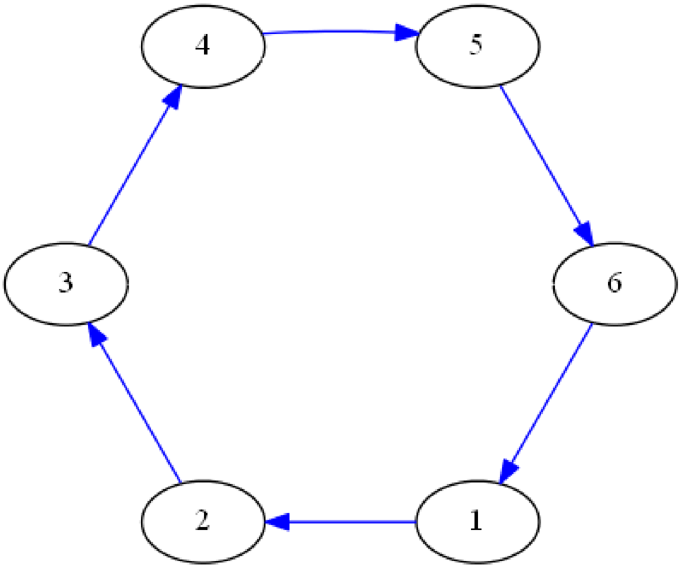
Given a digraph $D = (V, E)$, a cycle cover of D is a subset $C \subseteq E$, such that the induced digraph of C contains each vertex in V, and each vertex in the induced subgraph lies on exactly one cycle [7].

Permutation Parity

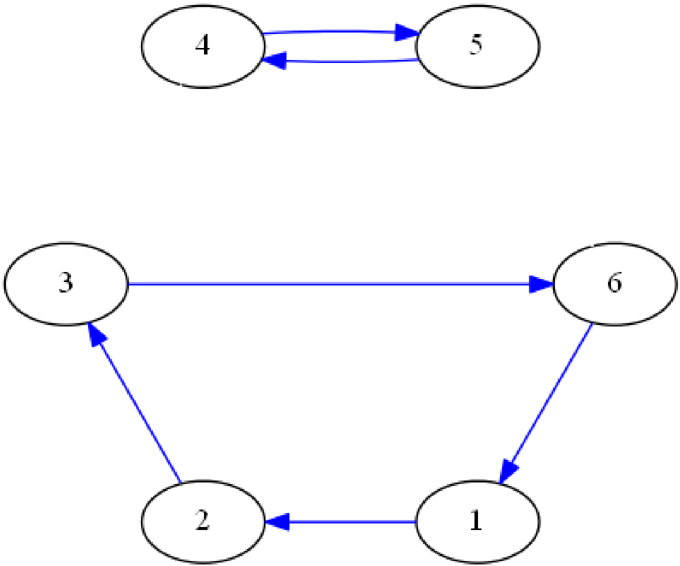
A cycle cover (permutation) is odd if it contains an odd number of even cycles.



Odd Cycle Cover



Even Cycle Cover



Permanents

The permanent of an $n \times n$ matrix, M, is defined as:

$$per(M) = \sum_{\pi \in S_n} \prod_{i=1}^{n} M_{i,\pi(i)},$$

- Determinant Similarities
- Differences
- Computational Complexity
- Counting with Permanents

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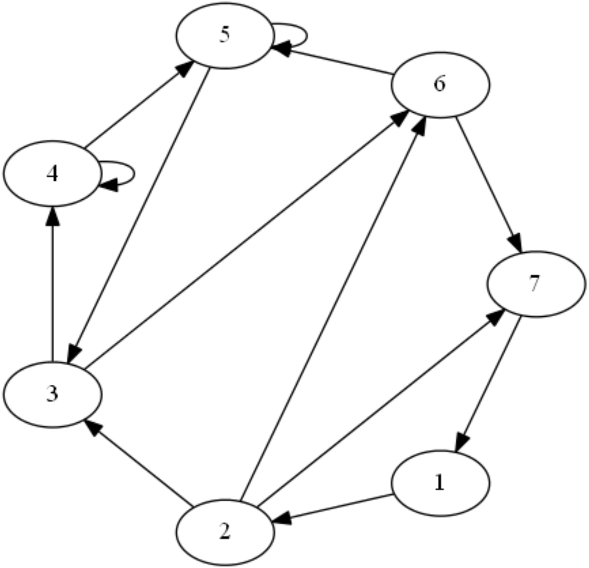
Labeled Digraph



Adjacency Matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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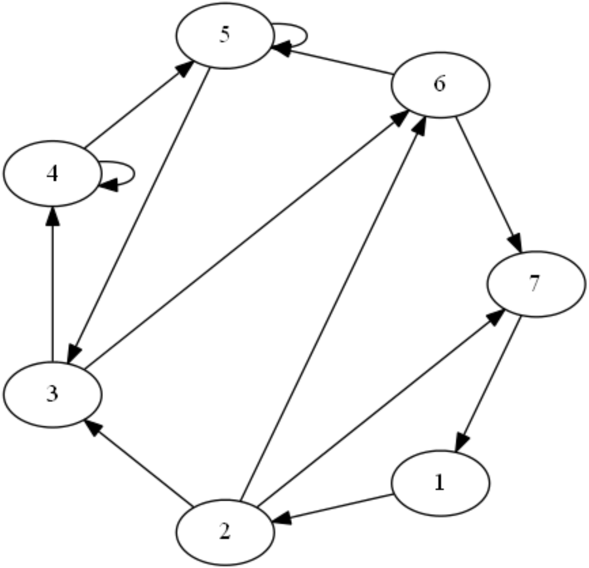

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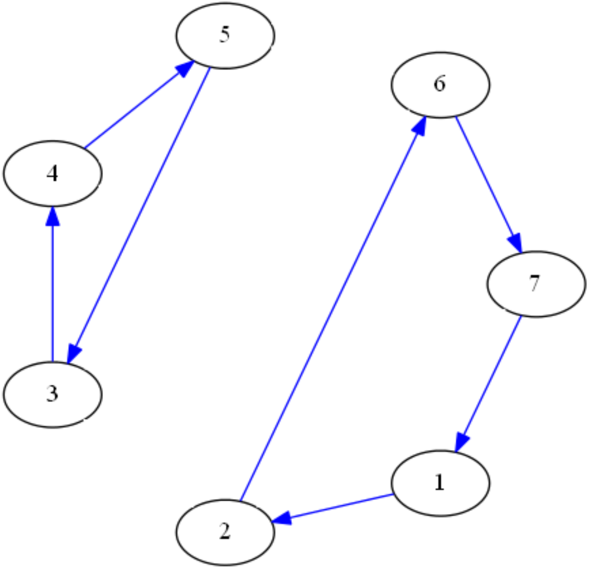
Cycle Covers



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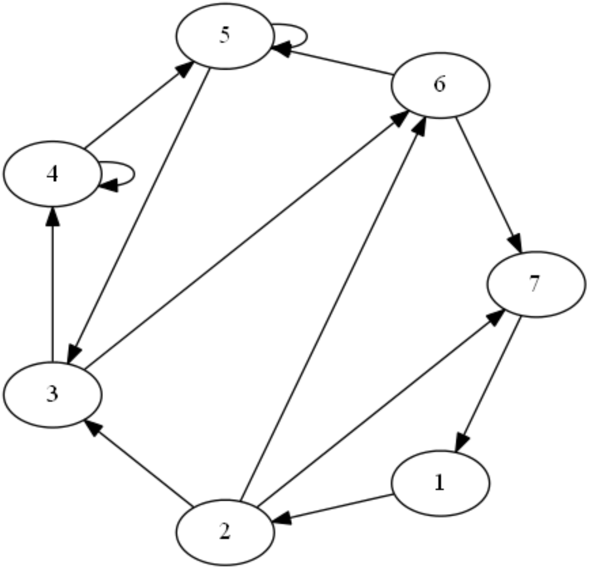

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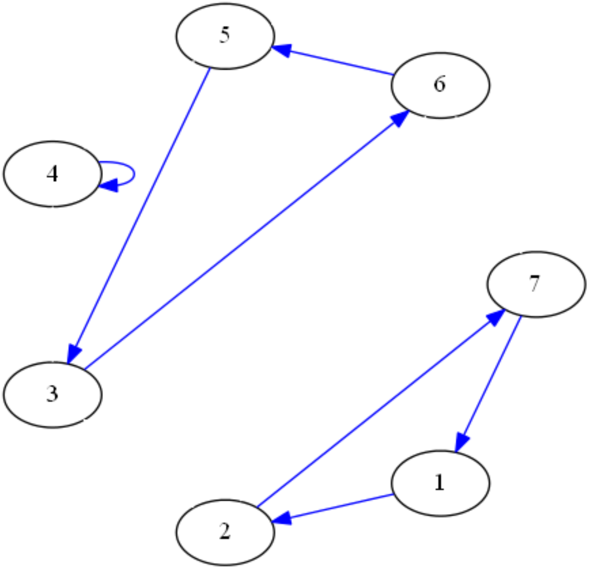
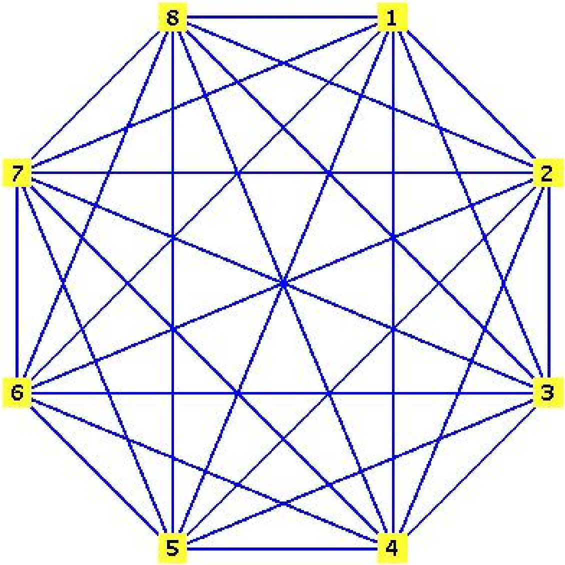
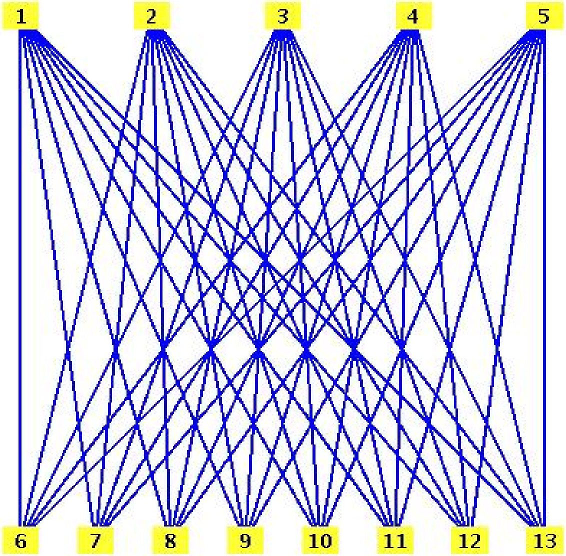


K_8



$K_{5,8}$



Simple Graphs

Graph	Rearrangements	With Stays
P_n	$0, 1, 0, 1, 0...$	f_n
C_n	$0, 1, 2, 4, 2, 4...$	$l_n + 2 = f_n + f_{n-2} + 2$
K_n	$D(n)$	$n!$
$K_{n,n}$	$(n!)^2$	$\sum_{i=0}^{n} \left[(n)_i\right]^2$
$K_{m,n}$ with $m \leq n$	0	$\sum_{i=0}^{m} (m)_i (n)_i$

Bipartite Graphs Theorem

Theorem

Let $G = (\{U, V\}, E)$ *be a bipartite graph. The number of rearrangements on* G *is equal to the square of the number of perfect matchings on* G.

Bipartite Graphs Proof

Proof.

Sketch.
Construct a bijection between pairs of perfect matchings on G and cycle covers on $\overleftrightarrow{G}$. WLOG select two perfect matchings of G, m_1 and m_2. For each edge, (u_1, v_1) in m_1 place a directed edge in the cycle cover from u_1 to v_1. Similarly, for each edge, (u_2, v_2) in m_2 place a directed edge in the cycle cover from v_2 to u_2. Since m_1 and m_2 are perfect matchings, by construction, each vertex in the cycle cover has in–degree and out–degree equal to 1.
Given a cycle cover C on $\overleftrightarrow{G}$ construct two perfect matchings on G by taking the directed edges from vertices in U to vertices in V separately from the directed edges from V to U. Each of these sets of (undirected) edges corresponds to a perfect matching by the definition of cycle cover and the bijection is complete. □

$P_2 \times G$ Theorem

Theorem

The number of rearrangements on a bipartite graph G, when the markers on G are permitted to remain on their vertices, is equal to the number of perfect matchings on $P_2 \times G$.

$P_2 \times G$ Proof

Proof.

Sketch.
Observe that $P_2 \times G$ is equivalent to two identical copies of G where each vertex is connected to its copy by a single edge (P_2). To construct a bijection between these two sets of objects, associate a self–loop in a cycle cover with an edge between a vertex and its copy in the perfect matching. Since the graph is bipartite, the remaining cycles in the cycle cover can be decomposed into matching edges from U to V and from V to U as in the previous theorem.

□

Seating Rearrangements with Stays

- Applying the previous theorem to the original problem of seating rearrangements gives that the number of rearrangements in a $m \times n$ classroom, where the students are allowed to remain in place or move is equal to the number of perfect matchings in $P_2 \times P_m \times P_n$. These matchings are equivalent to tiling a $2 \times m \times n$ rectangular prism with $1 \times 1 \times 2$ tiles.
- A more direct proof of this equivalence can be given by identifying each possible move type; up/down, left/right, or stay, with a particular tile orientation in space.

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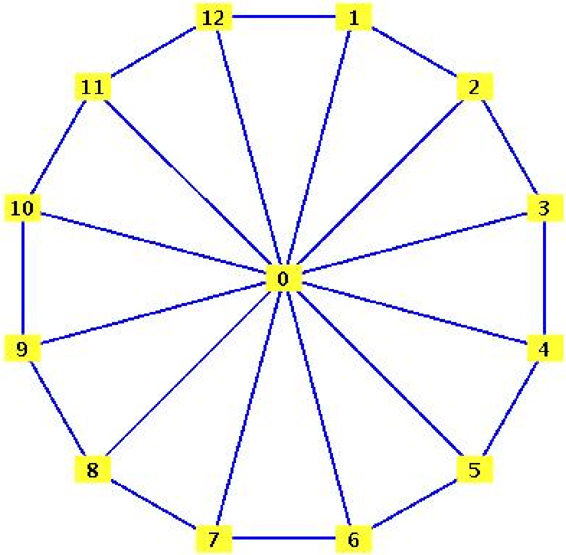
Wheel Graph Order 12



Wheel Graphs Rearrangements

The number of rearrangements on a wheel graph of order n is equal to n^2

- n odd
- Uniquely determined by the center vertex: $n \cdot n = n^2$
- n even
- Must create an odd cycle: $\frac{n}{2} \cdot 2n = n^2$

n	3	4	5	6	7	8	9	10	n
No stays	9	16	25	36	49	64	81	100	n^2
With stays	24	53	108	212	402	745	1356	2435	$\Rightarrow$

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Wheel Graph Rearrangements with Stays

The number of rearrangements on a wheel graph when the markers are permitted to either move or stay is equal to:

$$nf_{n+2} + f_n + f_{n-2} - 2n + 2$$

.

Condition on the behavior of the center marker:

- if it remains in place,
- $C_n = f_n + f_{n-2} + 2$
- if it moves to one of the n other vertices,
- $nf_{n-1} + 2n\sum_{k=2}^{n} f_{n-k} = nf_{n-1} + 2nf_n - 2n$

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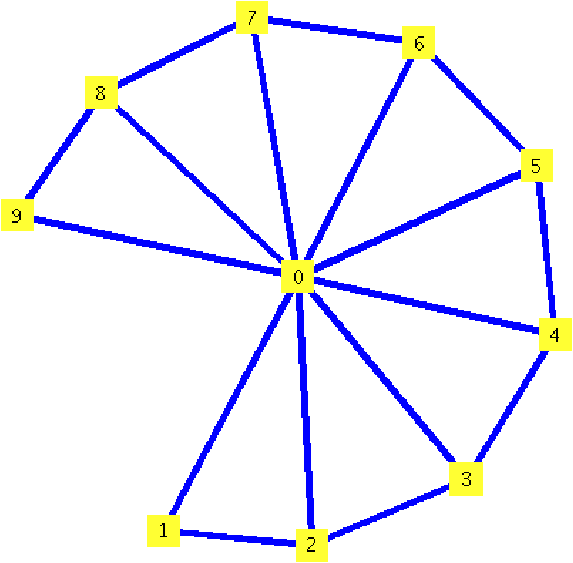
Flat Wheel of Order 9



Counting on Flat Wheels

- Without Stays: $\dfrac{n^2+2n+1}{4}$ (odd) or $\dfrac{n^2+2n}{4}$ (even).
- With Stays:

$$f_n + \sum_{l=1}^{n}\left[\left(f_{n-l}\sum_{j=0}^{l-2}[f_j]\right) + (f_{l-1}f_{n-l}) + \left(f_{l-1}\sum_{k=0}^{n-l-1}[f_k]\right)\right]$$

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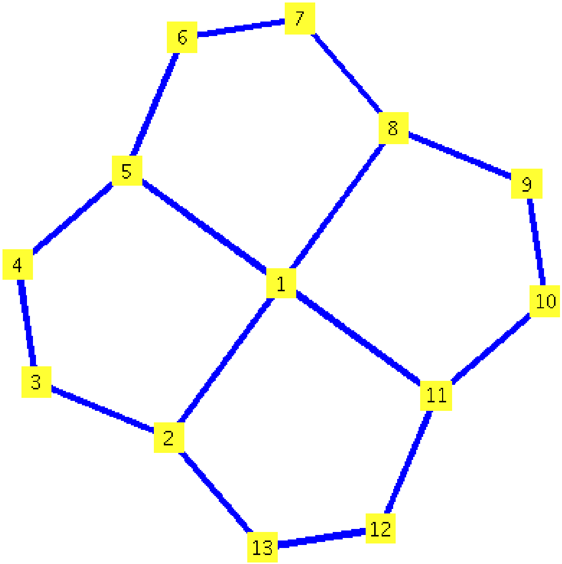
$k-$Wheel Graphs (Flower Graphs?)



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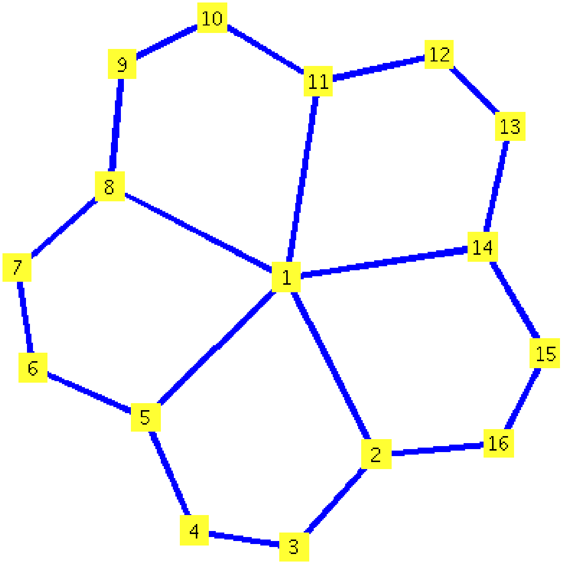

Counting on $k-$Wheel Graphs (Flower Graphs?)

- Without Stays: 0 (n and k are even) n^2 otherwise.
- With Stays:

$$l_{(k-2)n} + 2 + nf_{(k-2)n-1} + 2nf_{(n-2)k-(n-1)} + 2n\sum_{i=1}^{k-2} f_{(k-2)(n-i-1)-1}$$

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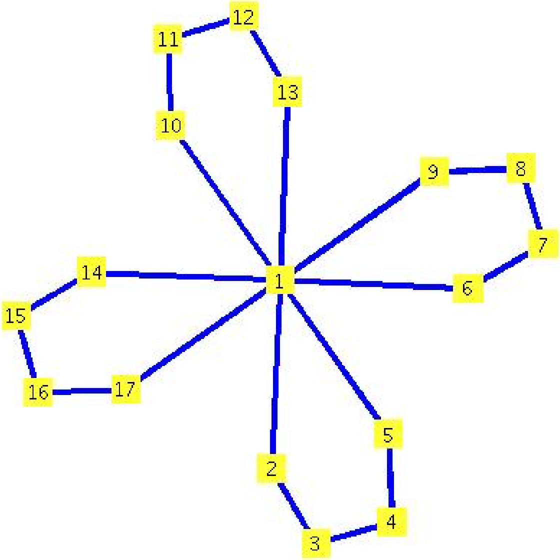
Dutch Windmill D_4^5



Counting on Dutch Windmills

- Without Stays: 0 (even) or $2m$ (odd)
- With Stays:

$$(f_{n-1})^m + 2m(f_{n-2}+1)(f_{n-1})^{m-1}$$

Counting on Dutch Windmills

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Hosoya Index of Trees

The Hosoya index is a topological invariant from computational chemistry that is equivalent to the total number of matchings on a graph. This index correlates with many physical properties of organic compounds, especially the alkanes (saturated hydrocarbons).

Theorem

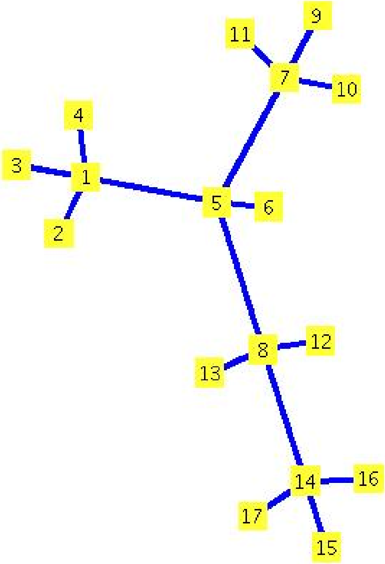
Let T be an $n-$tree with adjacency matrix $A(T)$. Then the Hosoya index of T is equal to $det(A(T)i + I_n)$

Hosoya Index Proof

Proof.

Sketch.
Since T is a tree there is a direct bijection between a given cycle cover on $\overleftrightarrow{T}$ with a self loop added to each vertex and a matching on T. Furthermore, $per(A(T) + I_n)$ counts these cycle covers. To see that $det(A(T)i + I_n) = per(A(T) + I_n)$ notice that each $2-$cycle and thus each even cycle counted in $det(A(T)i + I_n)$ has a weight of $i^2 = -1$, and thus that the weight of each cycle cover is equal to the sign of the permutation. □

Isopentane Example



$A(T)$

$$
per \left(\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix} \right) = 584
$$

$A(T)i + I_{17}$

$$
det\left(\begin{bmatrix}
1 & i & i & i & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
i & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
i & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
i & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 & 1 & i & i & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & i & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & i & 0 & 1 & 0 & i & i & i & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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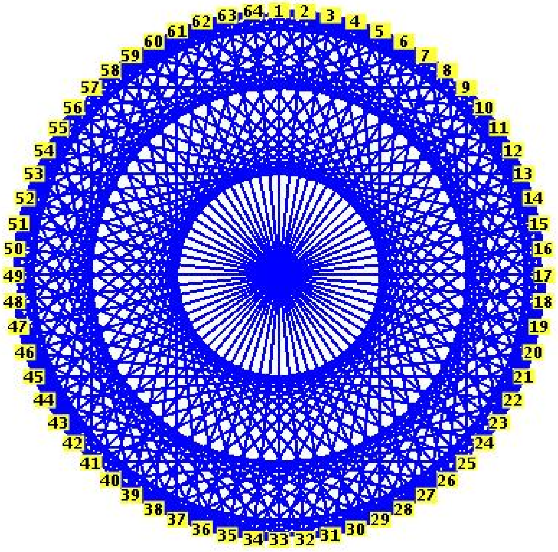
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 1
\end{bmatrix}\right) = 584
$$

Game Pieces

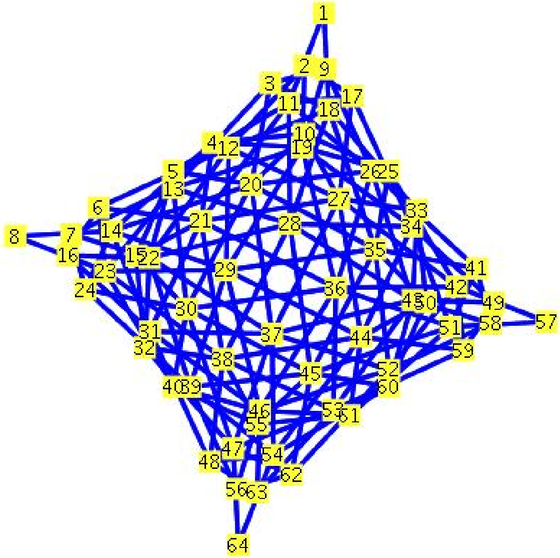
In order to generate well–motivated families of graphs, we turned to the following problem statement:

> *Consider an $m \times n$ chessboard along with mn copies of a particular game piece, one on each square. In how many ways can the pieces be rearranged if they must each make one legal move? Or at most one legal move? Can these rearrangement problems be solved with recurrence techniques?*

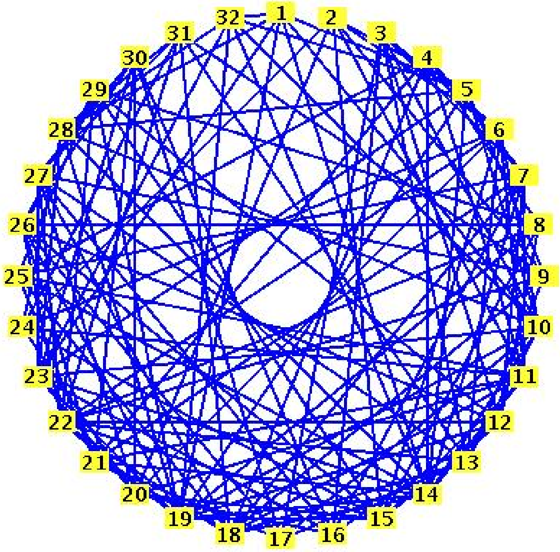
8×8 Rook Graph



8×8 Knight Graph



8×8 Bishop Graph



Fibonacci Relations

- $1 \times n$ Kings
- F_n
- $2 \times n$ Bishops
- F_n^2
- $2 \times 2n$ Knights
- F_n^4 or $F_n^2 * F_{n-1}^2$

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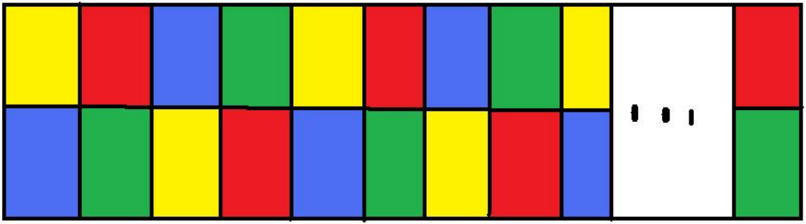
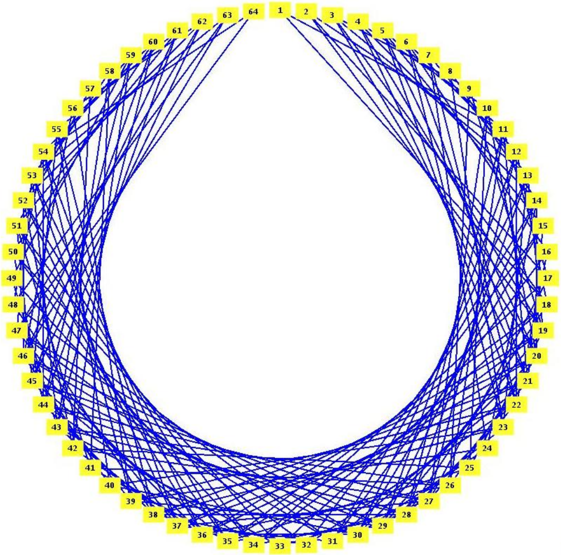
$2 \times 2n$ Knights



Knight Rearrangements



Knight's Tour

- 8×8 Knight's Tour
- 26,534,728,821,064 [4]
- 8×8 Knight Rearrangements
- 8,121,130,233,753,702,400

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LHCCRR Theorem

Theorem

On any rectangular $m \times n$ board B with m fixed, and a marker on each square, where the set of permissible movements has a maximum horizontal displacement, the number of rearrangements on B satisfies a linear, homogeneous, constant–coefficient recurrence relation as n varies.

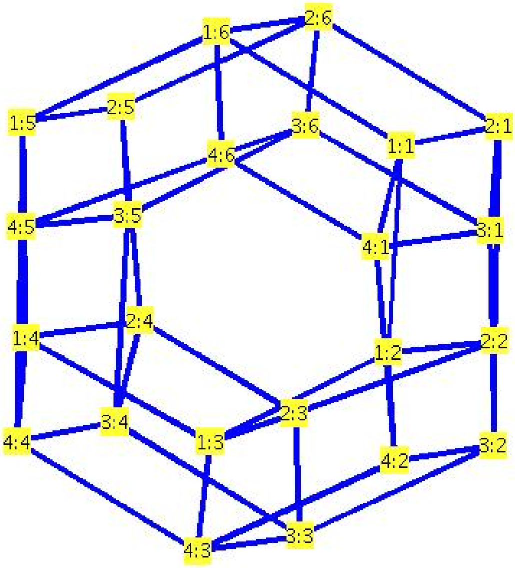
LHCCRR Proof

Proof.

Sketch.
Let d represent the maximum permissible horizontal displacement. Consider any set of marker movements that completes the first column. After all of the markers in the first column been moved, and other markers have been moved in to the first column to fill the remaining empty squares, any square in the initial $m \times d$ sub–rectangle may be in one of four states. Let S be the collection of all 4^{md} possible states of the initial $m \times d$ sub–rectangle, and let S^* represent the corresponding sequences counting the number of rearrangements of a board of length n beginning with each state as n varies. Finally, let a_n denote the sequence that describes the number of rearrangements on B as n varies.

For any board beginning with an element of S, consider all of possible sets of movements that "complete" the initial column. The resulting state is also in S, and has length $n - k$ for some k in $[1, d]$. Hence, the corresponding sequence can be expressed as a sum of elements in S^* with subscripts bounded below by $n - d$. This system of recurrences can be expressed as a linear, homogeneous, constant–coefficient recurrence relation in a_n either through the Cayley–Hamilton Theorem or by the successor operator matrix [4]. □

Other Surfaces



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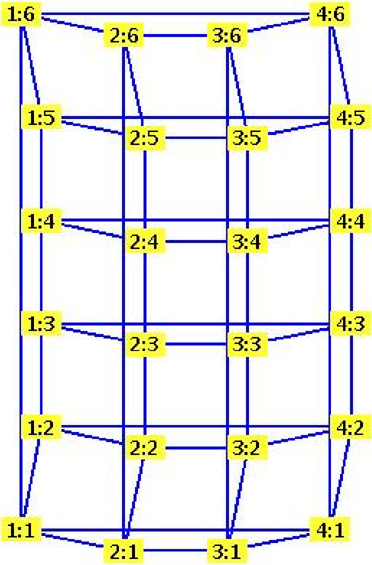

Dartmouth

Recurrence Orders

Although all of these problems lead to LHCCRR solutions, the growth rate between each instance of a particular set of movements makes it difficult to learn very much about the recurrences themselves.

- Kings (2, 3, 10, 27, 53, 100+)
- Knights (8, 27, lots)
- Verifying minimality :(

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- Verifying minimality :(

Can we bound the order?

- YES!!
-
- Can we do it easily? Sometimes.
-
- Does it impact the asymptotic analysis? Not even close

Example: Preliminaries

Lemma

The number of distinct Fibonacci tilings $\mathcal{S}(f_n)$ *of order* n *up to symmetry is equal to* $\frac{1}{2}(f_{2k} + f_{k+1})$ *when* $n = 2k$ *and* $\frac{1}{2}(f_{2k+1} + f_k)$ *when* $n = 2k + 1$.

Lemma

The number of endings with no consecutive 1×1 *tiles is equal to* P_{n+2}.

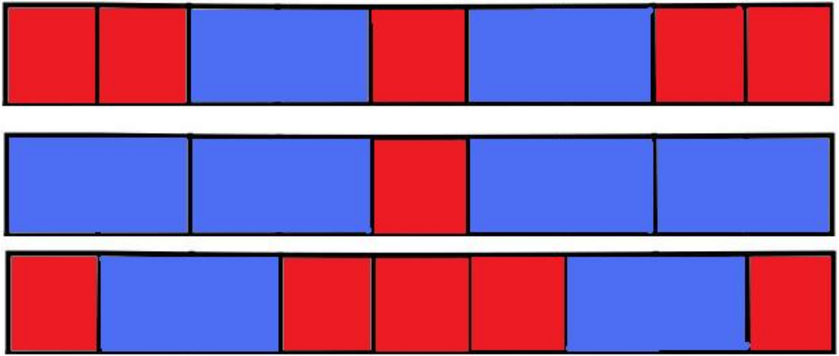
Lemma

The number of distinct Padovan tilings $\mathcal{S}(P_n)$*of order* n *up to symmetry is equal to* $\frac{1}{2}(P_{2k} + P_{k+2})$ *when* $n = 2k$ *and* $\frac{1}{2}(P_{2k+1} + P_{k-1})$ *when* $n = 2k + 1$.

Lemma Proofs

- The second lemma follows from a standard bijective double counting argument.
- The key to the first and third lemma is to realize that since every reflection of a particular tiling is another tiling we are over–counting by half, modulo the self–symmetric tilings. Adding these back in and a little parity bookkeeping completes the results.

Self–Symmetric Fibonacci Tilings



Example: Conclusion

Theorem

The minimal order of the recurrence relation for the number of tilings of a $k \times n$ rectangle with 1×1 and 2×2 squares is at most $\mathcal{S}(f_n) - \mathcal{S}(P_n) + 1$.

Table: Toy Example

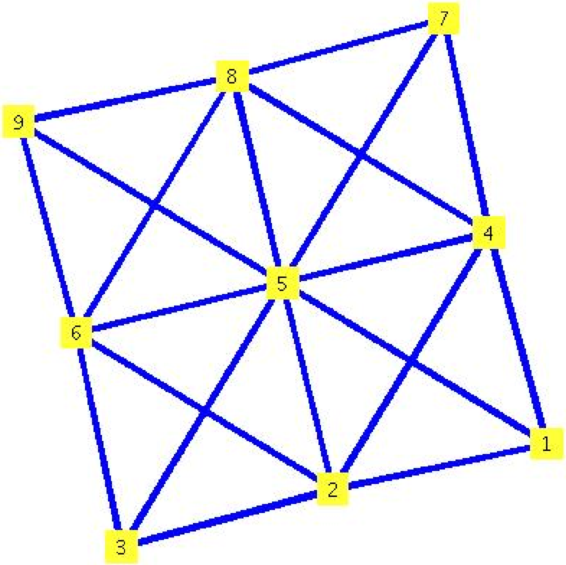
k	1	2	3	4	5	6	7	8	9	10
Upper Bound	1	4	9	25	64	169	441	1,156	3,025	7,921
$\mathcal{O}(T_n)$	1	2	2	3	4	6	8	14	19	32

Permanents and Tilings?

Theorem

There does not exist a simple graph whose (maximal) matchings can be placed into a one–to–one correspondence with the number of tilings of an $m \times n$ *rectangle with* 1×1 *and* 2×2 *squares, when* $m \geq 4$ *and* $n \geq 4$.

The Independent Set Graph



Proof Sketch

Proof.

Sketch. For any $m \times n$ rectangle construct a graph whose vertices represent the $(m-1)(n-1)$ possible center positions of a 2×2 square tile. Place an edge between any two vertices if a 2×2 square placed on the first vertex would intersect a 2×2 placed on the second. Call this graph $H_{m,n}$.

Notice, that an independent set (Maximal independent set if we place a self–loop on each vertex) of this graph is equivalent to a legitimate tiling of the rectangle. This gives us a one–to–one correspondence between tilings and independent sets. Thus, (maximal) matchings on the graph $G_{m,n}$ whose line graph is $H_{m,n}$ are in a similar correspondence to these tilings.

Unfortunately, there is no such graph $G_{m,n}$, by Beineke's forbidden minors theorem for line graphs [7]. Similarly, there is not directed or pseudo–graph with this property, although there is a family of hypergraphs, this does not help us count matchings. □

General $1 \times n$ Case

In the preceding example, knowing two $1 \times n$ cases allowed us to reduce the upper bound from $1,156$ to 10 without a significant amount of extra effort. Here we give an expression for all $1 \times n$ rectangular tilings, where the tiles in T are allowed to have multiple colors.

Notation

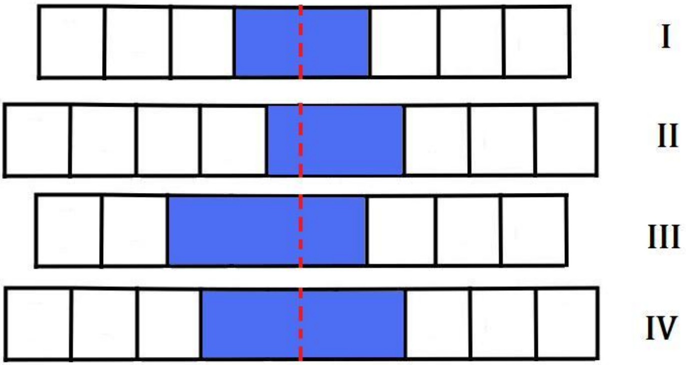
We begin by defining some convenient notation. Since we are covering boards of dimension $\{1 \times n | n \in \mathbb{N}\}$. Let $T = (a_1, a_2, a_3, \ldots)$, where a_m is the number of distinct colors of m–dominoes available. Then, T_n is the number of ways to tile a $1 \times n$ rectangle with colored dominoes in T. Connecting to our example, the Fibonacci numbers would be $T = (1, 1, 0, 0, 0, \ldots)$ while the Padovan numbers have $T = (0, 1, 1, 0, 0, 0, \ldots)$.

Coefficients

We also need to define a set of coefficients based on the parity of the domino length and the rectangle length.

$$c_j = \begin{cases} T_{n-\frac{j}{2}} & j \equiv n \equiv 0 \pmod 2 \\ 0 & j \equiv 0, n \equiv 1 \pmod 2 \\ 0 & j \equiv 1, n \equiv 0 \pmod 2 \\ T_{n-\frac{j-1}{2}} & j \equiv n \equiv 1 \pmod 2 \end{cases} \tag{1}$$

Coefficient Motivation



Complete Characterization of $1 \times n$ Tilings

Theorem

Let T be some set of colored k–dominoes, then the number of distinct tilings up to symmetry of a $1 \times n$ rectangle is equal to

$$\frac{1}{2}\left(T_n + \sum_{i=1}^{\infty} a_i c_i + \frac{T_{\frac{n}{2}}}{2} + \frac{(-1)^n T_{\frac{n}{2}}}{2}\right) \tag{2}$$

Lucas Tilings

It is natural to wonder if these methods could be adapted to give a similar formula for generalized Lucas tilings on a bracelet or necklace. Unfortunately, the complexity of the underlying symmetric group makes this a much more complex problem. Even in the simplest case we have:

Theorem

The number of distinct Lucas tilings of a $1 \times n$ bracelet up to symmetry is:

$$\sum_{i=0}^{\lceil \frac{n-1}{2} \rceil} \left[\frac{1}{n-i} \sum_{d|(i,n-1)} \varphi(d) \binom{\frac{n-i}{d}}{\frac{i}{d}} \right] \quad (3)$$

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Example 1

Example

The number of distinct rearrangements on a $2 \times n$ rectangle is

$$\frac{1}{4}\left(f_{2k}^2 + f_{2k} + 2f_k^2 + 2f_{k-1}^2\right) \tag{4}$$

when $n = 2k$ and

$$\frac{1}{4}\left(f_{2k+1}^2 + f_{2k+1} + 2f_k^2\right) \tag{5}$$

when $n = 2k + 1$

Example 2

Example

The number of distinct tilings of a $3 \times n$ rectangle with squares of size 1×1 and 2×2 is

$$\frac{1}{3}\left(2^{2n-1} + 2^n + 2^{n-1} + \frac{1+(-1)^n}{2}\right) \tag{6}$$

when n is odd, and

$$\frac{1}{3}\left(2^{2n} + 2^n + 2^{n-1} + 1\right) \tag{7}$$

when n is even.

Rectangle Symmetry

- $2 \times n$: Equivalent to the Fibonacci Tilings
- $3 \times n$: $\frac{1}{3}\left(2^{2n-1} + 2^n + 2^{n-1} + 1\right)$ when n is even and $\frac{1}{3}\left(2^{2n} + 2^n + \frac{1+(-1)^n}{2}\right)$ when n is odd.
- $4 \times n$ and $5 \times n$: Hideous, long generalized power sums with a mix of Fibonacci terms and eigenvalues of the original tiling recurrences
- Example $5 \times n$ odd:

$$\frac{1}{4}\left(2\left(\frac{\varphi^n + \overline{\varphi}^n}{\sqrt{5}}\right)\left(\left(\frac{\varphi^n + \overline{\varphi}^n}{\sqrt{5}}\right)^2\right)^n + (c_1\alpha + c_2\beta + c_3\gamma + c_4\delta)\left((c_1\alpha + c_2\beta + c_3\gamma + c_4\delta)^2\right)^n \cdots\right)$$

$\alpha =$

$$x = \frac{1}{2} - \frac{1}{2\sqrt{\frac{3}{17 - \frac{25}{\sqrt[3]{1333 - 108\sqrt{151}}} - \sqrt[3]{1333 - 108\sqrt{151}}}}} +$$

$$\frac{1}{2}\sqrt{\left(\frac{34}{3} + \frac{25}{3\sqrt[3]{1333 - 108\sqrt{151}}} + \frac{1}{3}\sqrt[3]{1333 - 108\sqrt{151}} - \frac{}{12\sqrt{\frac{3}{17 - \frac{25}{\sqrt[3]{1333 - 108\sqrt{151}}} - \sqrt[3]{1333 - 108\sqrt{151}}}}}\right)}$$

References

S. Aaronson: *A Linear-Optical Proof that the Permanent is #P Hard*, Proceedings of the Royal Society A,(2011).

M. Ahmadi and H. Dastkhzer: *On the Hosoya index of trees*, Journal of Optoelectronics and Advanced Materials, **13(9)**, (2011), 1122-1125

A. Balaban ed.: *Chemical Applications of Graph Theory*, Academic Press, London, 1976

L. Beineke and F. Harary: *Binary Matrices with Equal Determinant and Permanent*, Studia Scientiarum Academia Mathematica Hungarica, **1**, (1966), 179-183.

A. Benjamin and J. Quinn: *Proofs that Really Count*, MAA, Washington D.C., 2003.

N. Calkin, K. James, S. Purvis, S. Race, K. Schneider, and M. Yancey: *Counting Kings: Explicit Formulas, Recurrence Relations, and Generating Functions! Oh My!*, Congressus Numerantium **182** (2006), 41-51.

G. Chartrand, L. Lesniak, and P. Zhang: *Graphs & Digraphs Fifth Edition*, CRC Press, Boca Raton, 2011.

References

K. Collins and L. Krompart: *The Number of Hamiltonian Paths on a Rectangular Grid*, Discrete Math, **169**, (1997), 29-38.

T. Cormen, C. Leiserson, R. Rivest, and C. Stein: *Introduction To Algorithms Third Edition*, MIT Press, Cambridge, 2009.

W. Deskins: *Abstract Algebra*, Dover Publications, New York, 1995.

D. DeTemple and W. Webb: *The Successor Operator and Systems of Linear Recurrence Relations*, Private Communication.

R. Graham, D. Knuth, and O. Patashnik: *Concrete Mathematics*, Addison-Wesley, Reading, 1994.

I. Gutman, Z. Markovic, S.A. Markovic: *A simple method for the approximate calculation of Hosoya's index* , Chemical Physical Letters, **132(2)**, 1987, 139-142

F. Harary: *Determinants, Permanents and Bipartite Graphs*, Mathematics Magazine, **42(3)**,(1969), 146-148.

References

F. Harary: *Graph Theory and Theoretical Physics*, Academic P., New York, (1967).

S. Heubach: *Tiling an m–by–n Area with Squares of Size up to k–by–k* $(m \leq 5)$, Congrssus Numerantium **140**, (1999), 43-64.

R.Honsberger: *In Pólya's Footsteps*, MAA, New York, 1997.

H. Hosoya: *The Topological Index Z Before and After 1971*, Internet Electron. J. Mol. Des. **1**, (2002), 428-442.

H. Hosoya and I. Gutman: *Kekulè Structures of Hexagonal Chains–Some Unusual Connections*, J. Math. Chem. **44**, (2008), 559-568.

M. Huber: *Permanent Codes*, Duke University,
http://www.cmc.edu/pages/faculty/MHuber/Research/code/permanent_code/permanent_codes.html
, 2007.

References

M. Hudelson: *Vertex Topological Indices and Tree Expressions, Generalizations of Continued Fractions*, J. Math. Chem. **47**, (2010), 219-228.

A. Ilyichev, G. Kogan, V. Shevchenko: *Polynomial Algorithms for Computing the Permanents of some Matrices*, Discrete Mathematics and Applications, **7(4)** 1997, 413-417.

P. Kasteleyn: *The Statistics of Dimers on a Lattice: I. The Number of Dimer Arrangements on a Quadratic Lattice*, Physica, **27(12)**, (1961), 1209-1225.

R. Kennedy and C. Cooper: *Variations on a* 5×5 *Seating Rearrangement Problem*, Mathematics in College, **Fall-Winter**, (1993), 59-67.

G. Kuperberg: *An Exploration of the Permanents-Determinant Method*, Electronic Journal of Combinatorics, **5**, (1998), R46: 1-34.

J. H. van Lint and R. M. Wilson: *A Course in Combinatorics*, Cambridge University Press, Cambridge, 2001.

N. Loehr: *Bijective Combinatorics*, CRC Press, Boca Raton, 2011.

References

P. Lundow: *Computation of matching polynomials and the number of 1-factors in polygraphs*, Research Reports Umeå, **12**, (1996)

M. Marcus and H. Minc: *Permanents*, Amer. Math. Monthly, **72**, (1965), 577-591.

B. Mckay: *Combinatorial Data*, http://cs.anu.edu.au/ bdm/data/digraphs.html.

B. McKay: *Knight's Tours of an* 8×8 *Chessboard*, Technical Report TR-CS-97-03, Department of Computer Science, Australian National University, (1997).

OEIS Foundation Inc.: *The On-Line Encyclopedia of Integer Sequences*, http://oeis.org, (2012).

T. Otake, R. Kennedy, and C. Cooper: *On a Seating Rearrangement Problem*, Mathematics and Informatics Quarterly, **52**, (1996), 63-71.

C.Pinter: *A Book of Abstract Algebra*, Dover Publications, New York, 2010.

References

N. Robertson, P. D. Seymour, and R. Thomas: *Permanents, Pfaffian Orientations, and Even Directed Circuits*, Ann. Math. **150**, (1999), 929-975.

D. Rouvray ed.: *Computational Chemical Graph Theory*, Nova Science Publishers, New York, 1990.

J. Sellers: *Domino Tilings and Products of Fibonacci and Pell Numbers*, Journal of Integer Sequences, **5**, (2002), 02.1.2 1-6.

G. Shilov and R. Silverman: *Linear Algebra*, Dover Publications, New York, 1977.

A. Slomson: *An Introduction to Combinatorics*, Chapman and Hall, London, 1991.

H. Temperley and M. Fischer: *Dimer Problem in Statistical Mechanics-An Exact Result*, Philosophical Magazine, **6(68)**, (1961), 1061-1063.

R. Tichy and S Wagner: *Extremal Problems for Topological Indices in Combinatorial Chemistry*, Journal of Computational Biology, **12(7)**, (2005), 1004-1013

References

N. Trinajstic: *Chemical Graph Theory Volume I*, CRC Press, Boca Raton, 1983

N. Trinajstic: *Chemical Graph Theory Volume II*, CRC Press, Boca Raton, 1983

M. van de Wiel and A. Di Bucciánico: *Fast Computation of the Exact Null Distribution of Spearman's ρ and Page's L Statistic for Samples With and Without Ties*, Eindhoven University of Technology, Netherlands, 1998.

L. valiant: *The Complexity of Computing the Permanent*, Theoretical Computer Science, **8(2)**, (1979), 189-201.

V. Vazirani and M. Yannakakis: *Pfaffian Orientations, 0/1 Permanents, and Even Cycles in Directed Graphs*, Lecture Notes in Computer SCience: Automata, Languages, and Programming, (1998), 667-681.

W. Webb: *Matrices with Forbidden Submatrics*, Private Communication.

W. Webb, N. Criddle, and D. DeTemple: *Combinatorial Chessboard Tilings*, Congressus Numereratium **194** (2009), 257262.

That's all...

Thank You.