Counting Combinatorial Rearrangements, Tilings With Squares, and Symmetric Tilings

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Outline

Overview

- Rearrangements on Graphs
- Rearrangements on Chessboards
- Tiling $m \times n$ Rectangles with Squares
- Symmetric Tilings

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Motivation

Recurrence Relations

Particularly Linear Homogeneous Constant–Coefficient Recurrence Relations (LHCCRR). The equivalence of these relations with rational generating functions and generalized power sums as vector spaces over $\mathbb{C}$ is a very useful tool.

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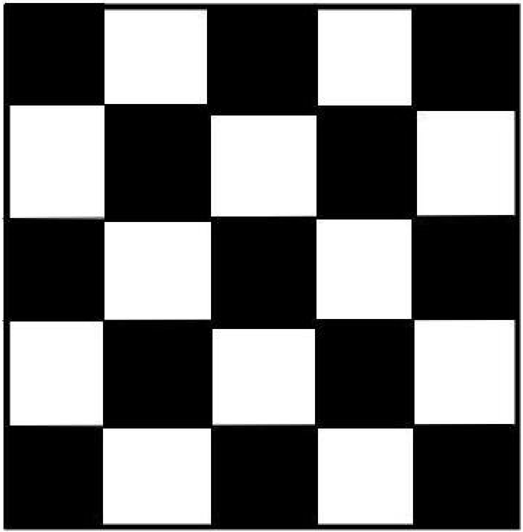
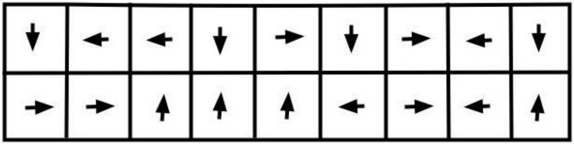
Original Problem (Honsberger)

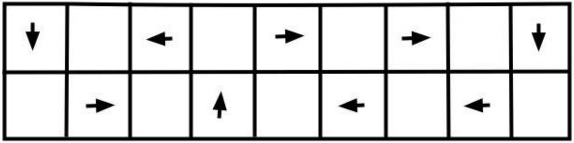
A classroom has 5 rows of 5 desks per row. The teacher requests each pupil to change his seat by going either to the seat in front, the one behind, the one to his left, or the one to his right (of course not all these options are possible to all students). Determine whether or not his directive can be carried out.

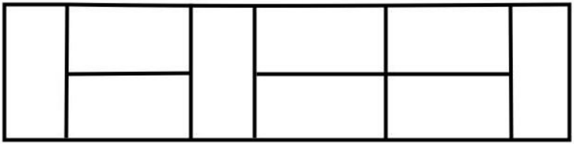
Original Problem



Seating Rearrangements and Tilings







Arbitrary Graphs

In order to count rearrangements on arbitrary graphs, we constructed the following problem statement:

Problem

Given a graph, place a marker on each vertex. We want to count the number of legitimate "rearrangements" of these markers subject to the following rules:

- *Each marker must move to an adjacent vertex.*
- *After all of the markers have moved, each vertex must contain exactly one marker.*

To permit markers to **either** *remain on their vertex or move to an adjacent vertex, add a self-loop to each vertex (forming a pseudograph)*

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Digraphs

With this problem statement we can describe these rearrangements mathematically as follows:

- Given a graph G, construct $\overleftrightarrow{G}$, by replacing each edge in G with a two directed edges (one in each orientation).
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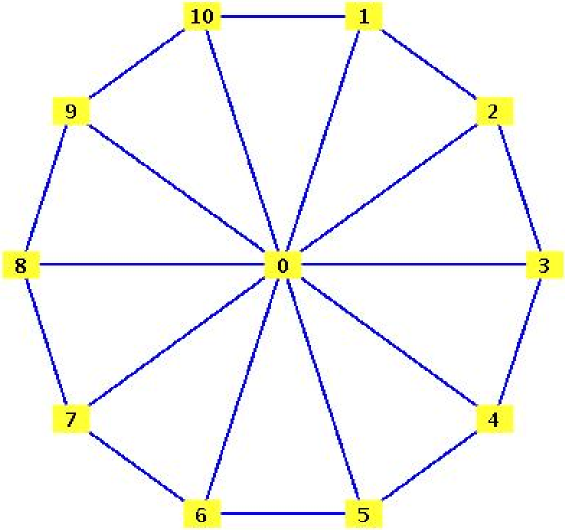

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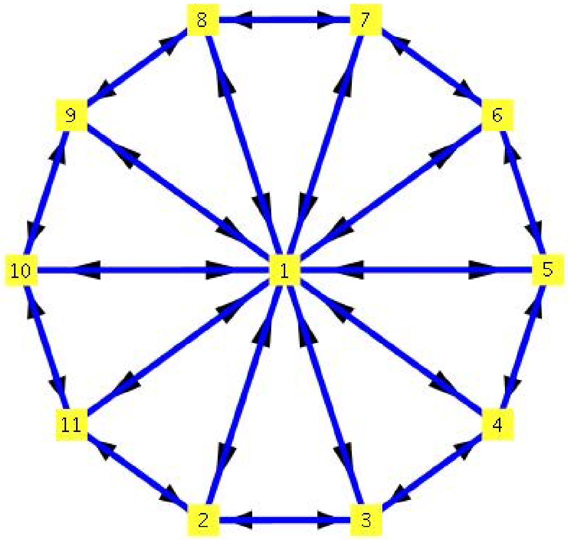


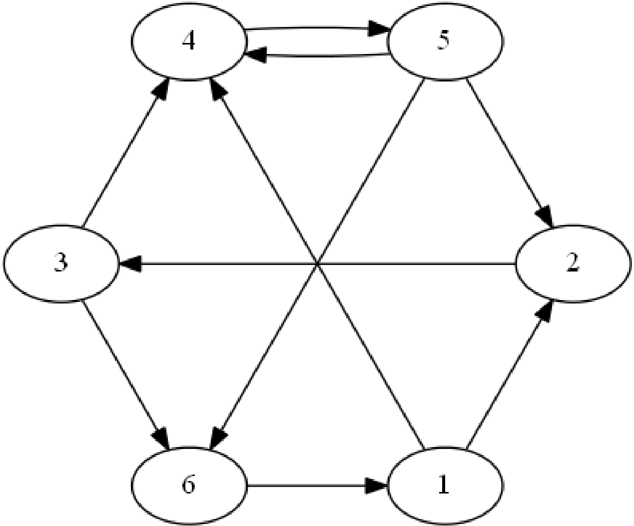
Cycle Covers

Definition

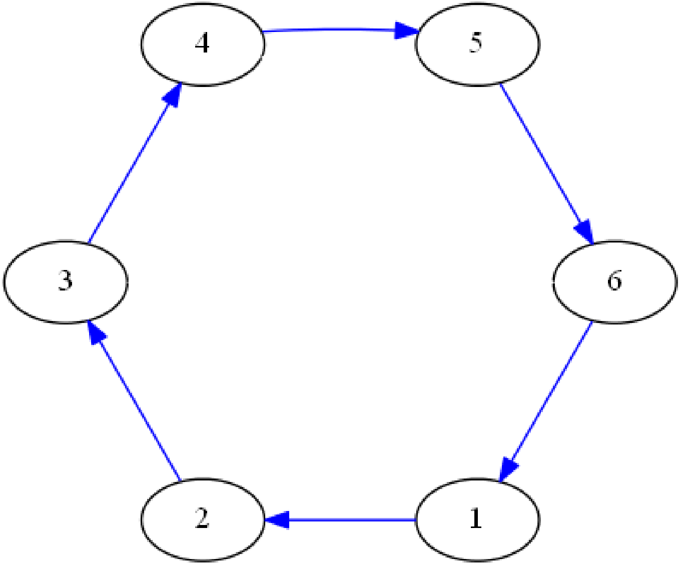
Given a digraph $D = (V, E)$, a cycle cover of D is a subset $C \subseteq E$, such that the induced digraph of C contains each vertex in V, and each vertex in the induced subgraph lies on exactly one cycle [7].

Permutation Parity

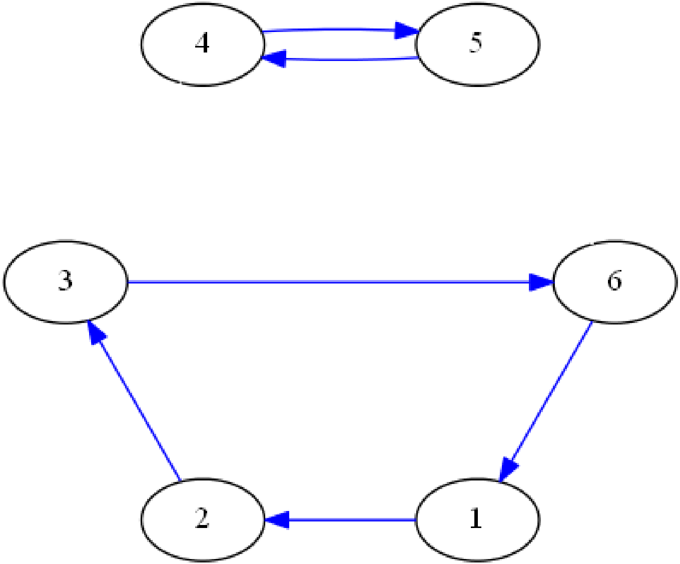
A cycle cover (permutation) is odd if it contains an odd number of even cycles.



Odd Cycle Cover



Even Cycle Cover



Permanents

The permanent of an $n \times n$ matrix, M, is defined as:

$$per(M) = \sum_{\pi \in S_n} \prod_{i=1}^{n} M_{i,\pi(i)},$$

- Determinant Similarities
- Differences
- Computational Complexity
- Counting with Permanents

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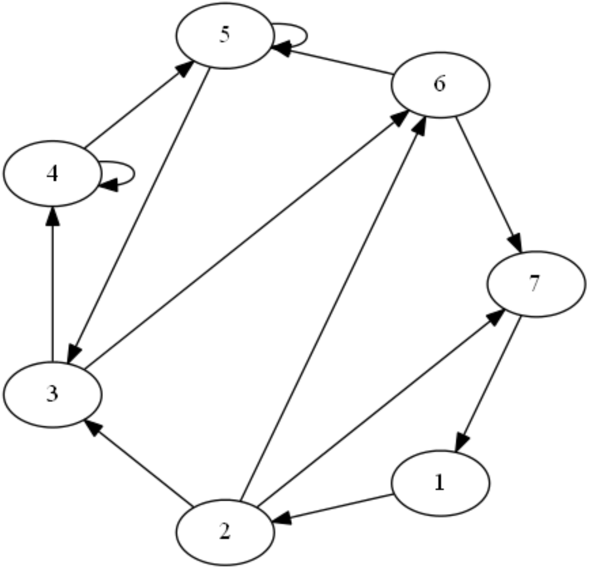
Labeled Digraph



Adjacency Matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$per(A) = 2$$

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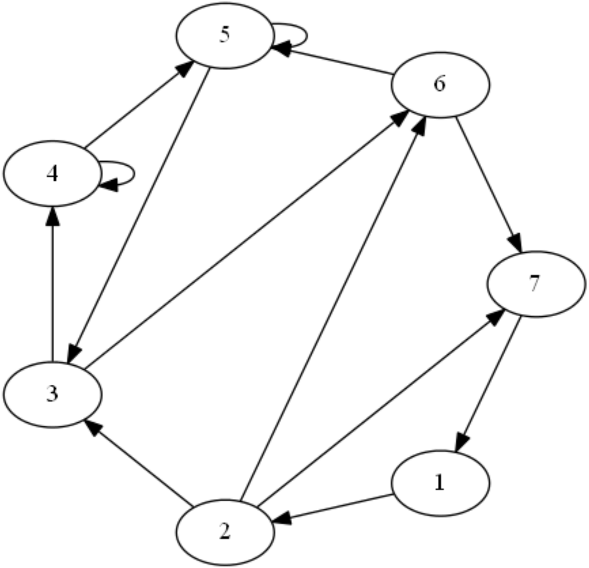

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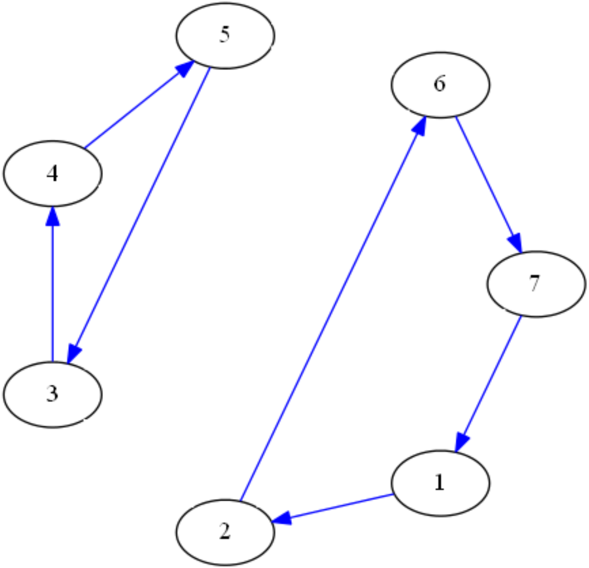

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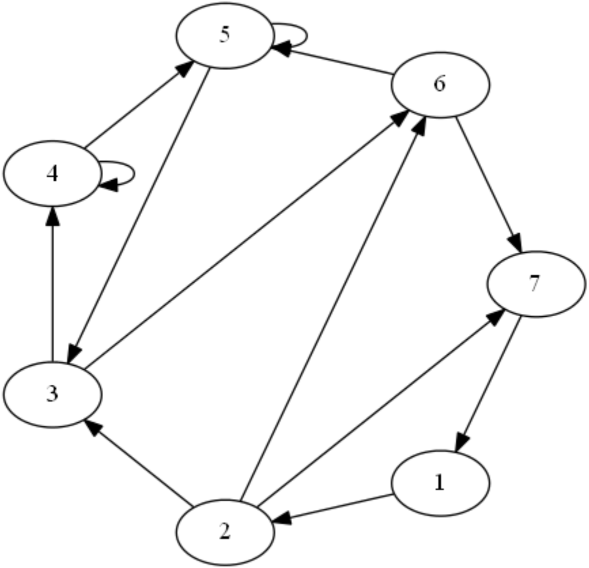

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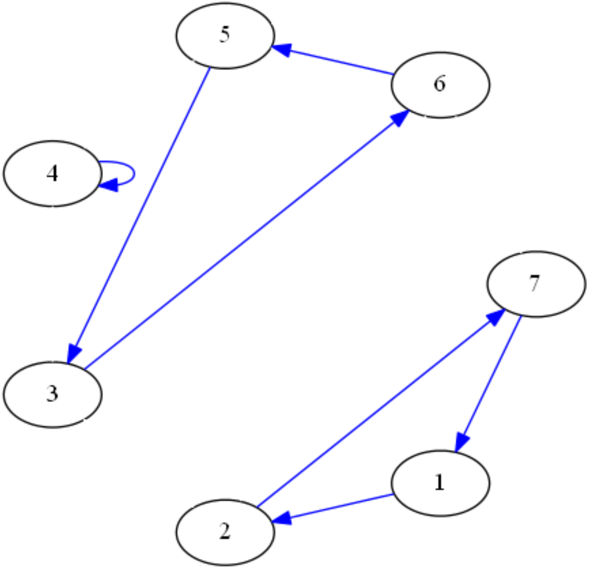
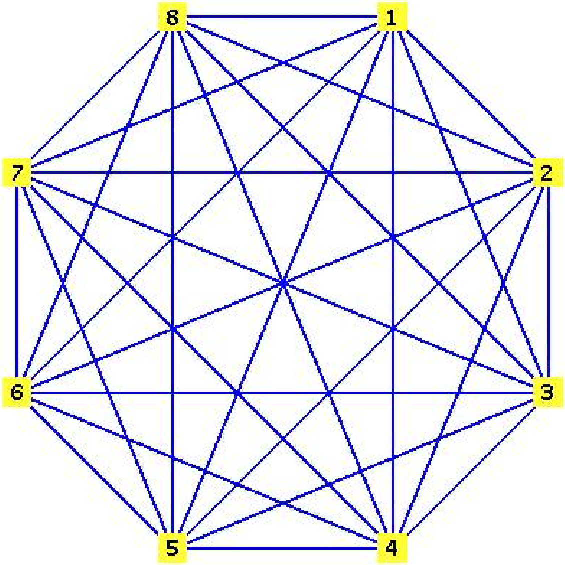
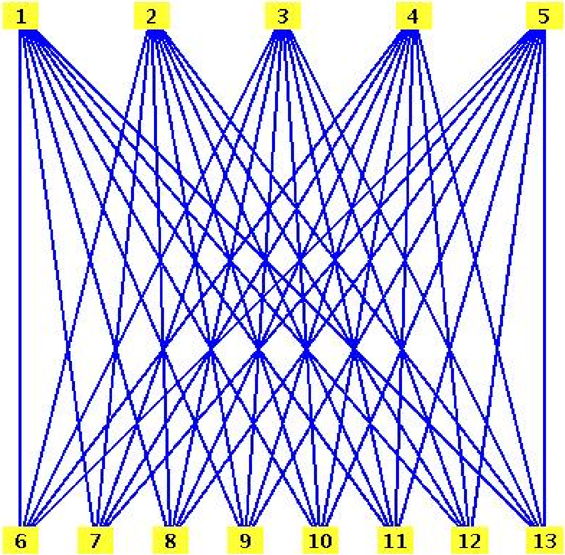


K_8



$K_{5,8}$



Simple Graphs

Graph	Rearrangements	With Stays
P_n	$0, 1, 0, 1, 0 \ldots$	f_n
C_n	$0, 1, 2, 4, 2, 4 \ldots$	$l_n + 2 = f_n + f_{n-2} + 2$
K_n	$D(n)$	$n!$
$K_{n,n}$	$(n!)^2$	$\sum_{i=0}^{n} [(n)_i]^2$
$K_{m,n}$ with $m \leq n$	0	$\sum_{i=0}^{m} (m)_i (n)_i$

Bipartite Graphs Theorem

Theorem

Let $G = (\{U, V\}, E)$ be a bipartite graph. The number of rearrangements on G is equal to the square of the number of perfect matchings on G.

Bipartite Graphs Proof

Proof.

Sketch.
Construct a bijection between pairs of perfect matchings on G and cycle covers on $\overleftrightarrow{G}$. WLOG select two perfect matchings of G, m_1 and m_2. For each edge, (u_1, v_1) in m_1 place a directed edge in the cycle cover from u_1 to v_1. Similarly, for each edge, (u_2, v_2) in m_2 place a directed edge in the cycle cover from v_2 to u_2. Since m_1 and m_2 are perfect matchings, by construction, each vertex in the cycle cover has in–degree and out–degree equal to 1.

Given a cycle cover C on $\overleftrightarrow{G}$ construct two perfect matchings on G by taking the directed edges from vertices in U to vertices in V separately from the directed edges from V to U. Each of these sets of (undirected) edges corresponds to a perfect matching by the definition of cycle cover and the bijection is complete. □

$P_2 \times G$ Theorem

Theorem

The number of rearrangements on a bipartite graph G, when the markers on G are permitted to remain on their vertices, is equal to the number of perfect matchings on $P_2 \times G$.

$P_2 \times G$ Proof

Proof.

Sketch.
Observe that $P_2 \times G$ is equivalent to two identical copies of G where each vertex is connected to its copy by a single edge (P_2). To construct a bijection between these two sets of objects, associate a self–loop in a cycle cover with an edge between a vertex and its copy in the perfect matching. Since the graph is bipartite, the remaining cycles in the cycle cover can be decomposed into matching edges from U to V and from V to U as in the previous theorem. □

Seating Rearrangements with Stays

- Applying the previous theorem to the original problem of seating rearrangements gives that the number of rearrangements in a $m \times n$ classroom, where the students are allowed to remain in place or move is equal to the number of perfect matchings in $P_2 \times P_m \times P_n$. These matchings are equivalent to tiling a $2 \times m \times n$ rectangular prism with $1 \times 1 \times 2$ tiles.
- A more direct proof of this equivalence can be given by identifying each possible move type; up/down, left/right, or stay, with a particular tile orientation in space.

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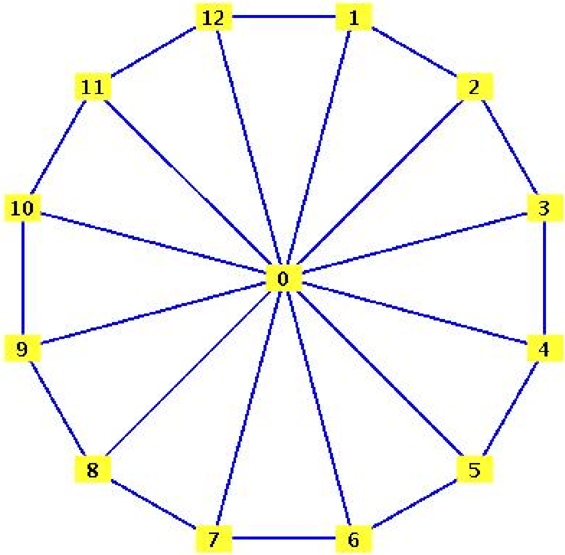
Wheel Graph Order 12



Wheel Graphs Rearrangements

The number of rearrangements on a wheel graph of order n is equal to n^2

- n odd
- Uniquely determined by the center vertex: $n \cdot n = n^2$
- n even
- Must create an odd cycle: $\frac{n}{2} \cdot 2n = n^2$

n	3	4	5	6	7	8	9	10	n
No stays	9	16	25	36	49	64	81	100	n^2
With stays	24	53	108	212	402	745	1356	2435	$\Rightarrow$

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Wheel Graph Rearrangements with Stays

The number of rearrangements on a wheel graph when the markers are permitted to either move or stay is equal to:

$$nf_{n+2} + f_n + f_{n-2} - 2n + 2$$

.

Condition on the behavior of the center marker:

- if it remains in place,
- $C_n = f_n + f_{n-2} + 2$
- if it moves to one of the n other vertices,
- $nf_{n-1} + 2n\sum_{k=2}^{n} f_{n-k} = nf_{n-1} + 2nf_n - 2n$

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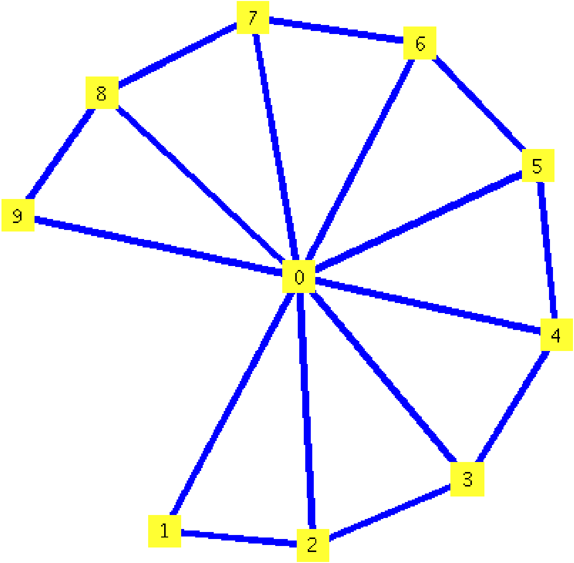
Flat Wheel of Order 9



Counting on Flat Wheels

- Without Stays: $\dfrac{n^2+2n+1}{4}$ (odd) or $\dfrac{n^2+2n}{4}$ (even).
- With Stays:

$$f_n + \sum_{l=1}^{n}\left[\left(f_{n-l}\sum_{j=0}^{l-2}[f_j]\right) + (f_{l-1}f_{n-l}) + \left(f_{l-1}\sum_{k=0}^{n-l-1}[f_k]\right)\right]$$

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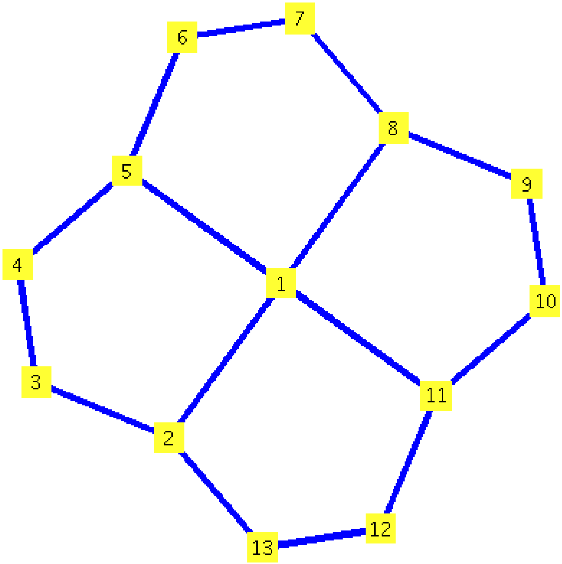
$k-$Wheel Graphs (Flower Graphs?)



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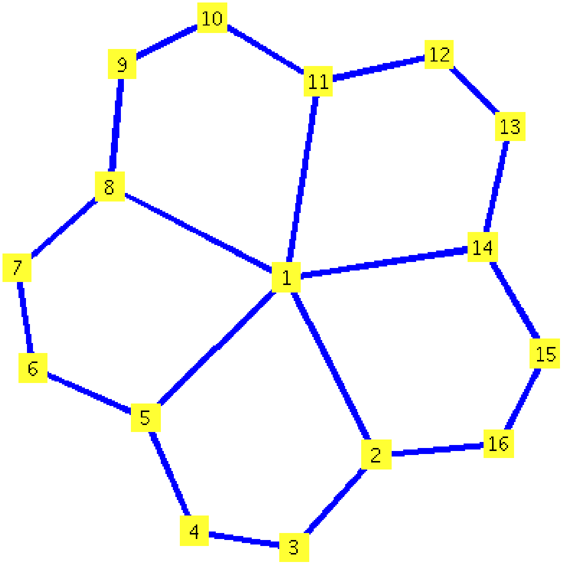


Counting on $k-$Wheel Graphs (Flower Graphs?)

- Without Stays: 0 (n and k are even) n^2 otherwise.
- With Stays:

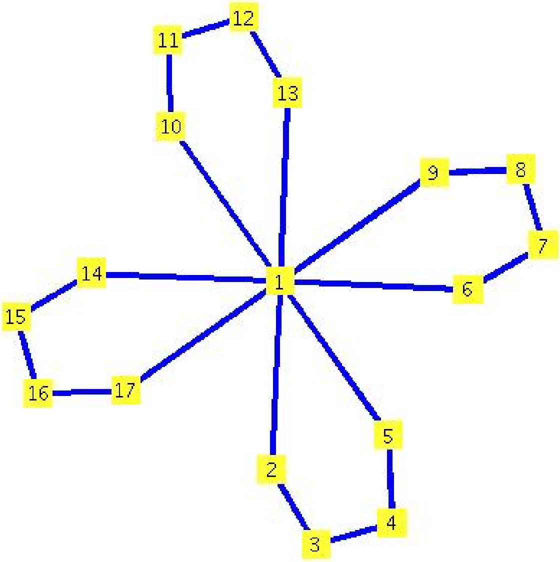
$$l_{(k-2)n} + 2 + nf_{(k-2)n-1} + 2nf_{(n-2)k-(n-1)} + 2n\sum_{i=1}^{k-2} f_{(k-2)(n-i-1)-1}$$

Counting on $k-$Wheel Graphs (Flower Graphs?)

- Without Stays: 0 (n and k are even) n^2 otherwise.
- With Stays:

$$l_{(k-2)n} + 2 + nf_{(k-2)n-1} + 2nf_{(n-2)k-(n-1)} + 2n\sum_{i=1}^{k-2} f_{(k-2)(n-i-1)-1}$$

Dutch Windmill D_4^5



Counting on Dutch Windmills

- Without Stays: 0 (even) or $2m$ (odd)
- With Stays:

$$(f_{n-1})^m + 2m(f_{n-2}+1)(f_{n-1})^{m-1}$$

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Rearrangement Recurrences

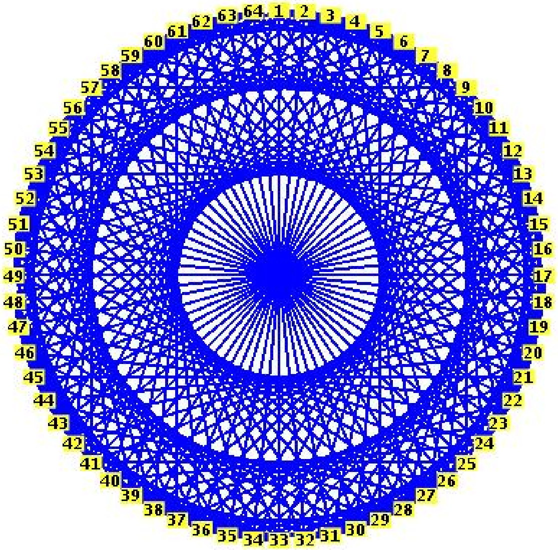
Although this model provides many interesting problems to count, in terms of generating problem specific recurrences it is fairly inefficient. Thus, we turned to another family of similar problems in order to attempt to study these sequences.

Game Pieces

In order to generate well–motivated families of graphs, we turned to the following problem statement:

> *Consider an $m \times n$ chessboard along with mn copies of a particular game piece, one on each square. In how many ways can the pieces be rearranged if they must each make one legal move? Or at most one legal move? Can these rearrangement problems be solved with recurrence techniques?*

8 × 8 Rook Graph



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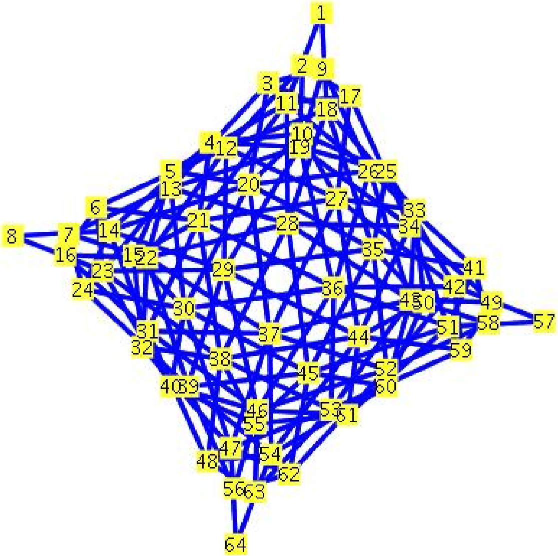
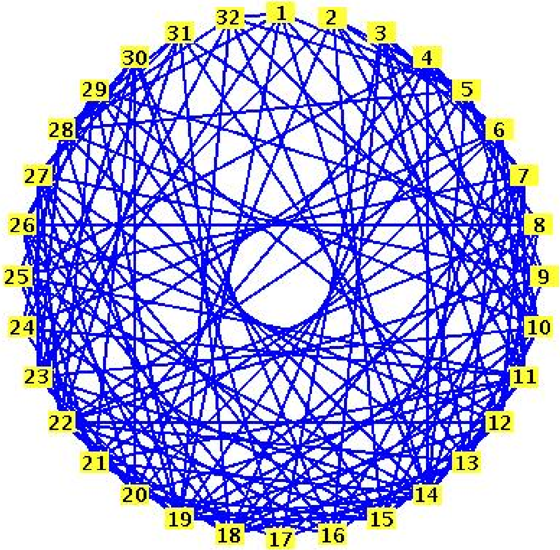
8 × 8 Knight Graph



8 × 8 Bishop Graph



Fibonacci Relations

- $1 \times n$ Kings
- F_n
- $2 \times n$ Bishops
- F_n^2
- $2 \times 2n$ Knights
- F_n^4 or $F_n^2 * F_{n-1}^2$

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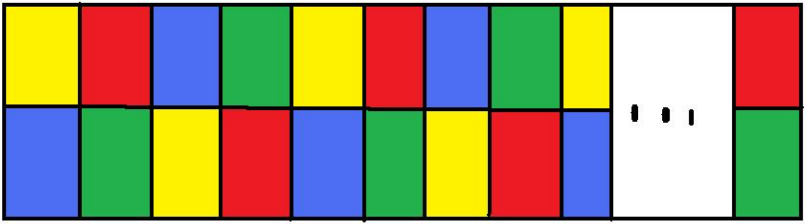
$2 \times 2n$ Knights



LHCCRR Theorem

Theorem

On any rectangular $m \times n$ board B with m fixed, and a marker on each square, where the set of permissible movements has a maximum horizontal displacement, the number of rearrangements on B satisfies a linear, homogeneous, constant–coefficient recurrence relation as n varies.

LHCCRR Proof

Proof.

Sketch.
Let d represent the maximum permissible horizontal displacement. Consider any set of marker movements that completes the first column. After all of the markers in the first column been moved, and other markers have been moved in to the first column to fill the remaining empty squares, any square in the initial $m \times d$ sub–rectangle may be in one of four states. Let S be the collection of all 4^{md} possible states of the initial $m \times d$ sub–rectangle, and let S^* represent the corresponding sequences counting the number of rearrangements of a board of length n beginning with each state as n varies. Finally, let a_n denote the sequence that describes the number of rearrangements on B as n varies.

For any board beginning with an element of S, consider all of possible sets of movements that "complete" the initial column. The resulting state is also in S, and has length $n - k$ for some k in $[1, d]$. Hence, the corresponding sequence can be expressed as a sum of elements in S^* with subscripts bounded below by $n - d$. This system of recurrences can be expressed as a linear, homogeneous, constant–coefficient recurrence relation in a_n either through the Cayley–Hamilton Theorem or by the successor operator matrix [4]. □

Recurrence Orders

Although all of these problems lead to LHCCRR solutions, the growth rate between each instance of a particular set of movements makes it difficult to learn very much about the recurrences themselves.

- Kings (2, 3, 10, 27, 53, 100+)
- Knights (8, 27, lots)
- Verifying minimality :(

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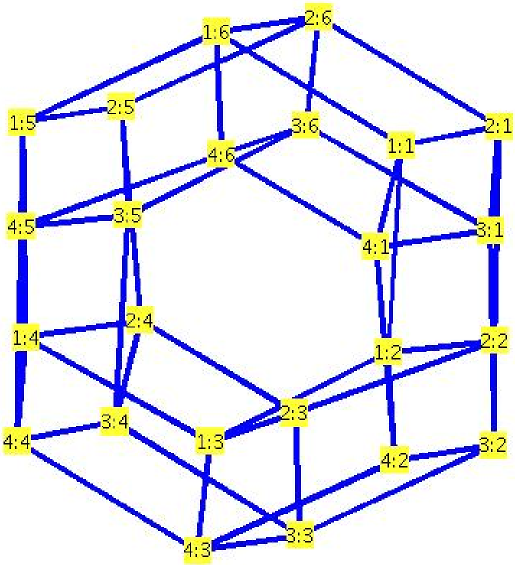
Other Surfaces



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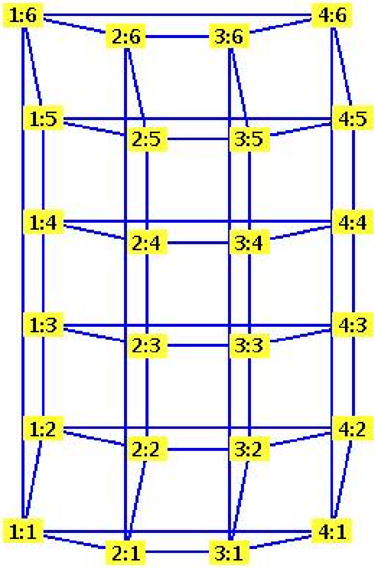
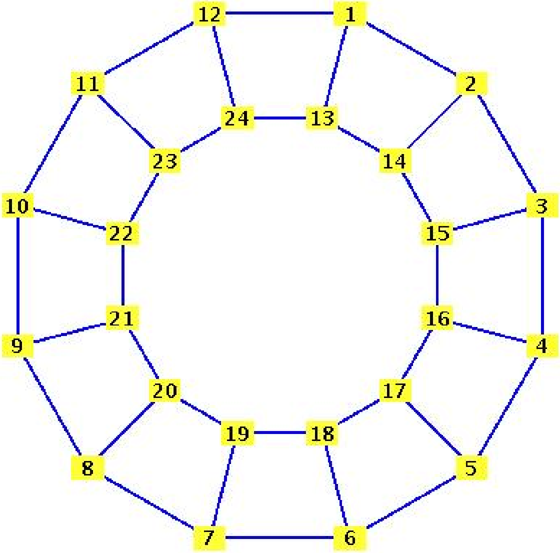


Prism Graph of Order 12



Prism Graphs

The number of rearrangements on a prism graph of order n is equal to $(l_n+2)^2$ if n is even and $l_{2n}+2$ if n is odd.

- n is even.
- The graph is bipartite and isomorphic to $C_n \times P_2$. Hence, the number of rearrangements is equal to the square of the number of rearrangements on C_n with stays permitted.
- n is odd.
- There is a bijection between pairs of Lucas tilings of length n and prism graph rearrangements where at least one marker moves between rows. The only uncounted rearrangements are the four where each marker remains in its original row. Thus, we have

$$l_n^2 + 4 = (l_n^2 + 2) = l_{2n} + 2$$

n	3	4	5	6	7	8	n
No stays	20	81	125	400	845	2401	$l_{2n}+2$ | $(l_n+2)^2$
With stays	82	272	890	3108	11042	39952	

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Prism Rearrangements with Stays

The number of rearrangements with stays on a prism graph of order n is given by the following generalized power sum:

$$6 + 4(-1)^n + \left(2 + \sqrt{3}\right)^n + \left(2 - \sqrt{3}\right)^n + \left(1 + \sqrt{2}\right)^n + \left(1 - \sqrt{2}\right)^n$$

Unfortunately, our method is inefficient in its approach (12×12 matrix with repeated eigenvalues).

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Shift Gears

Tiling problems have traditionally been one of the best ways to motivate recurrence relations combinatorially [5]. Generalized Fibonacci tilings with a more algebraic flavor can be found in Benjamin and Quinn's book [5].

Problem Introduction

Problem Statement:

Given an $m \times n$ rectangular board and an unlimited number of square tiles of various, previously defined dimension, in how many ways can the board be tiled?

- Heubach "Basic" Blocks [2]
- Calkin et al. Matrix Methods [6]

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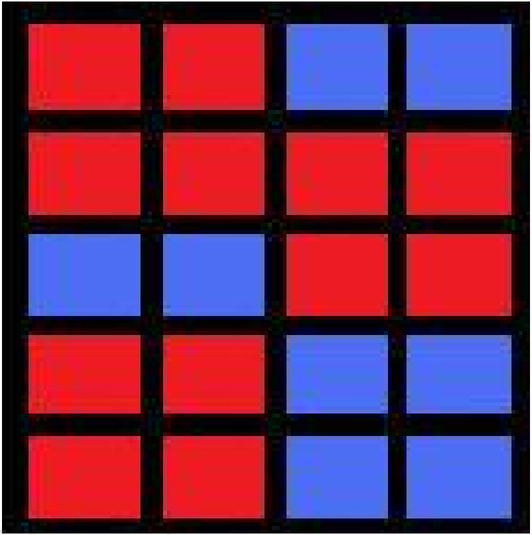
Rectangle Tiling Example



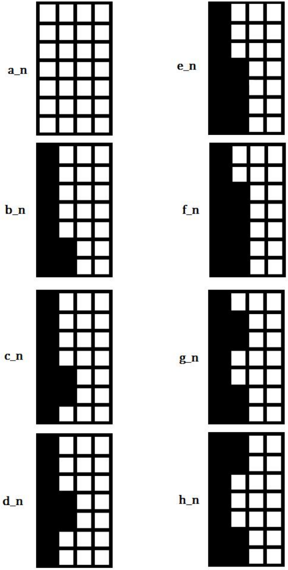
Generalization

A previous proof due to Dr. Webb and Dr. DeTemple guarantees that every problem of this type satisfies some LHCCRR, and provides the basic road–map of how to compute such a relation [7]. Similarly, their recently introduced the successor–operator matrix provides a convenient tool for doing the actual computations [4].

Examples

- $7 \times n$ with 1×1 and 2×2
- $5 \times n$ with all square sizes
- $4 \times 4 \times n$ with all cube sizes

$7 \times n$ Rectangle Endings



$7 \times n$ Rectangles

$$
\begin{aligned}
a_n &= a_{n-1} + 5a_{n-2} + 2b_{n-1} + 2c_{n-1} + \\
&\quad 2d_{n-1} + 2e_{n-1} + 4f_{n-1} + 2g_{n-1} + h_{n-1} \\
b_n &= a_{n-1} + b_{n-1} + c_{n-1} + 2d_{n-1} + e_{n-1} + 2f_{n-1} \\
c_n &= a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1} + e_{n-1} \\
d_n &= a_{n-1} + 2b_{n-1} + c_{n-1} + g_{n-1} + h_{n-1} \\
e_n &= a_{n-1} + b_{n-1} + c_{n-1} \\
f_n &= a_{n-1} + b_{n-1} \\
g_n &= a_{n-1} + d_{n-1} \\
h_n &= a_{n-1} + 2d_{n-1}
\end{aligned}
$$

$7 \times n$ Rectangles

$$M = \begin{bmatrix} E^2 - E - 5 & -2E & -2E & -2E & -2E & -4E & -2E & -E \\ -1 & E-1 & -1 & -2 & -1 & -2 & 0 & 0 \\ -1 & -1 & E-1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -2 & -1 & E & 0 & 0 & -1 & -1 \\ -1 & -1 & -1 & 0 & E & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & E & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & E & 0 \\ -1 & 0 & 0 & -2 & 0 & 0 & 0 & E \end{bmatrix}$$

$7 \times n$ Rectangles

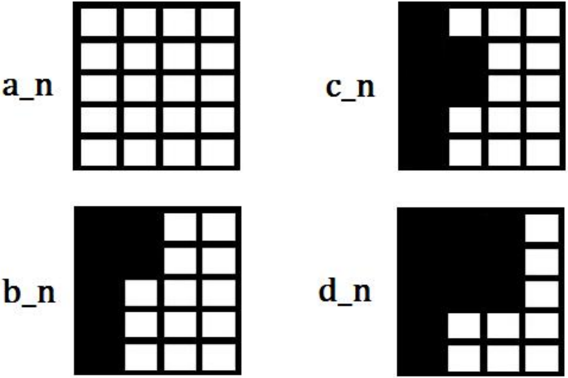
A Characteristic Polynomial:

$$det(M) = E^9 - 3E^8 - 30E^7 + 17E^6 + 138E^5 - 85E^4 - 116E^3 + 42E^2 + 32E$$

Recurrence Relation:

$$a_n = 3a_{n-1} + 30a_{n-2} - 17a_{n-3} - 138a_{n-4} + 85a_{n-5} + 116a_{n-6} - 42a_{n-7} - 32a_{n-8}$$

$5 \times n$ Rectangle Endings



$5 \times n$ Rectangles System

$$
\begin{aligned}
a_n &= a_{n-1} + 3a_{n-2} + a_{n-3} + 2a_{n-4} + a_{n-5} \\
&+ 2b_{n-1} + 2c_{n-1} + 2_{n-2} + 2d_{n-1} \\
b_n &= a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1} \\
c_n &= a_{n-1} + b_{n-1} \\
d_n &= a_{n-2} + c_{n-1}
\end{aligned}
$$

$5 \times n$ Rectangles Matrix

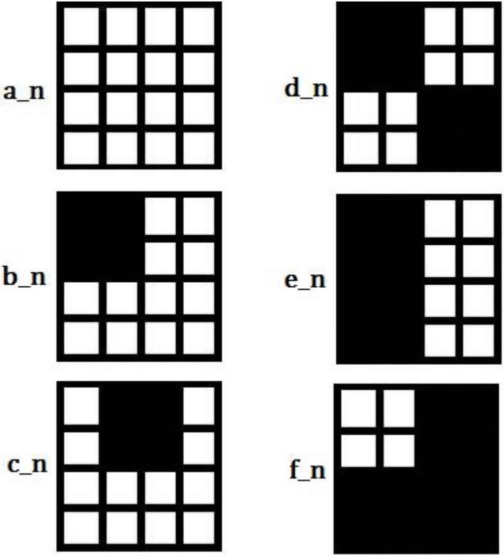
$$\begin{bmatrix} E^5 - E^4 - 3E^3 - E^2 - 2E - 1 & -2E^4 & -2E^4 - 2E^3 & -2E^4 \\ -1 & E - 1 & -1 & -1 \\ -1 & -1 & E & 0 \\ -1 & 0 & -E & E^2 \end{bmatrix}$$

$5 \times n$ Rectangles Recurrence

Recurrence Relation:

$$a_n = 2a_{n-1} + 7a_{n-2} + 6a_{n-3} - a_{n-4} - 6a_{n-5} - a_{n-7} - 2a_{n-8}$$

$4 \times 4 \times n$ Prism Endings



$4 \times 4 \times n$ Prism System

$$
\begin{aligned}
a_n &= a_{n-1} + 8a_{n-2} + 4a_{n-3} + a_{n-4} \\
&+ 4b_{n-1} + 4c_{n-1} + 2d_{n-1} + 4e_{n-1} + 12f_{n-1} \\
b_n &= a_{n-1} + 3b_{n-1} + 2c_{n-1} + d_{n-1} + 2e_{n-1} + 3f_{n-1} \\
c_n &= a_{n-1} + 2b_{n-1} + c_{n-1} + e_{n-1} \\
d_n &= a_{n-1} + 2b_{n-1} + d_{n-1} \\
e_n &= a_{n-1} + 2b_{n-1} + c_{n-1} + e_{n-1} \\
f_n &= a_{n-1} + b_{n-1}
\end{aligned}
$$

$4 \times 4 \times n$ Prism Matrix

$$\begin{bmatrix} E^4 - E^3 - 8E^2 - 4E - 1 & -4E^3 & -4E^3 & -2E^3 - 4E^3 & -12E^3 & \\ -1 & E-3 & -2 & -1 & -2 & -3 \\ -1 & -2 & E-1 & 0 & -1 & 0 \\ -1 & -2 & 0 & E-1 & 0 & 0 \\ -1 & -2 & -1 & 0 & E-1 & 0 \\ -1 & -1 & 0 & 0 & 0 & E \end{bmatrix}$$

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$4 \times 4 \times n$ Prism Recurrence

Recurrence Relation:

$$a_n = 7a_{n-1} + 28a_{n-2} - 123a_{n-3} + 18a_{n-4} + 84a_{n-5} + 20a_{n-6} + a_{n-7} - 2a_{n-8}$$

Recurrence Order Upper Bound

The order of the recurrence that counts the number of tilings of an m by n rectangle with 1×1 and 2×2 tiles as m is fixed and n varies is bounded above by

$$\frac{1}{2}(f_{2k} + f_{k+1} - P_{2k} - P_{k+2}) + 1$$

when $m = 2k$ and

$$\frac{1}{2}(f_{2k+1} + f_k - P_{2k+1} - P_{k-1}$$

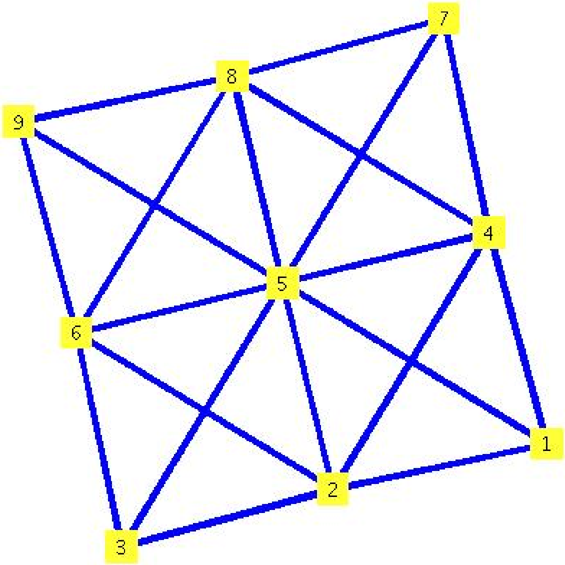
(where P_j is the j^{th} Padavan number).

Permanents and Tilings?

Theorem

There does not exist a simple graph whose (maximal) matchings can be placed into a one–to–one correspondence with the number of tilings of an $m \times n$ *rectangle with* 1×1 *and* 2×2 *squares, when* $m \geq 4$ *and* $n \geq 4$.

The Independent Set Graph



Proof Sketch

Proof.

Sketch. For any $m \times n$ rectangle construct a graph whose vertices represent the $(m-1)(n-1)$ possible center positions of a 2×2 square tile. Place an edge between any two vertices if a 2×2 square placed on the first vertex would intersect a 2×2 placed on the second. Call this graph $H_{m,n}$.
Notice, that an independent set (Maximal independent set if we place a self–loop on each vertex) of this graph is equivalent to a legitimate tiling of the rectangle. This gives us a one–to–one correspondence between tilings and independent sets. Thus, (maximal) matchings on the graph $G_{m,n}$ whose line graph is $H_{m,n}$ are in a similar correspondence to these tilings.
Unfortunately, there is no such graph $G_{m,n}$, by Beineke's forbidden minors theorem for line graphs [7]. Similarly, there is not directed or pseudo–graph with this property, although there is a family of hypergraphs, this does not help us count matchings. □

Recurrences

These bounded smaller order recurrences offer much more hope for learning about the recurrence properties themselves.

Pòlya Counting

- Burnside's Lemma
- Pòlya Counting
- Colorings of Geometric Objects

Fibonacci Symmetry

Here there is only one dimension, and only two elements in the symmetry group. We get

$$\frac{1}{2}(f_{2k} + f_{k+1})$$

when $n = 2k$ and

$$\frac{1}{2}(f_{2k+1} + f_k)$$

when $n = 2k + 1$.
It was originally the work on the previously discussed upper bound that motivated these problems.

Lucas Symmetry

Since Lucas tilings are circular we can consider both bracelets (which allow flipping) and necklaces (which do not). For necklaces we have that the number of tilings of a particular length n is:

$$\sum_{i=0}^{\lceil \frac{m-1}{2} \rceil} f(m-i, i)$$

where

$$f(a,b) = \frac{1}{a} \sum_{d|(a,b)} \varphi(d) \binom{\frac{a}{d}}{\frac{b}{d}}$$

Thus, we have this delightful expression [5]:

$$\sum_{i=0}^{\lceil \frac{m-1}{2} \rceil} \frac{1}{m-i} \sum_{d|(i,m-i)} \varphi(d) \binom{\frac{m-i}{d}}{\frac{i}{d}} = \frac{1}{m} \sum_{d|m} \varphi\left(\frac{m}{d}\right) [f_{d+1} + f_{d-1}]$$

Lucas Symmetry

- Double Sums $\leftrightarrow$ different number of dominoes
- Problems applying Pòlya methods

Lucas Symmetry

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Rectangle Symmetry

- $2 \times n$: Equivalent to the Fibonacci Tilings
- $3 \times n$: $\frac{1}{3}\left(2^{2n-1} + 2^n + 2^{n-1} + 1\right)$ when n is even and $\frac{1}{3}\left(2^{2n} + 2^n + \frac{1+(-1)^n}{2}\right)$ when n is odd.
- $4 \times n$ and $5 \times n$: Hideous, long generalized power sums with a mix of Fibonacci terms and eigenvalues of the original tiling recurrences
- Example $5 \times n$ odd:

$$\frac{1}{4}\left(2\left(\frac{\varphi^n + \overline{\varphi}^n}{\sqrt{5}}\right)\left(\left(\frac{\varphi^n + \overline{\varphi}^n}{\sqrt{5}}\right)^2\right)^n + (c_1\alpha + c_2\beta + c_3\gamma + c_4\delta)\left((c_1\alpha + c_2\beta + c_3\gamma + c_4\delta)^2\right)^n \cdots\right)$$

$\alpha =$

$$x = \frac{1}{2} - \frac{1}{2\sqrt{\frac{3}{17 - \frac{25}{\sqrt[3]{1333 - 108\sqrt{151}}} - \sqrt[3]{1333 - 108\sqrt{151}}}}} +$$

$$\frac{1}{2}\sqrt{\left(\frac{34}{3} + \frac{25}{3\sqrt[3]{1333 - 108\sqrt{151}}} + \frac{1}{3}\sqrt[3]{1333 - 108\sqrt{151}} - \frac{}{12\sqrt{\frac{3}{17 - \frac{25}{\sqrt[3]{1333 - 108\sqrt{151}}} - \sqrt[3]{1333 - 108\sqrt{151}}}}}\right)}$$

New Directions

- These problems get difficult very quickly (Lucas tilings, rectangles with squares, ...)
- Tilings with dominoes
- Kasteleyn's Identity, lots of ways to attack
- Other, simpler tiling problems (fixed tile orientations, squares and dominoes, ...)

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That's all...

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