Èdouard Lucas:

The theory of recurrent sequences is an inexhaustible mine which contains all the properties of numbers; by calculating the successive terms of such sequences, decomposing them into their prime factors and seeking out by experimentation the laws of appearance and reproduction of the prime numbers, one can advance in a systematic manner the study of the properties of numbers and their application to all branches of mathematics.

Matched Products and Stirling Numbers of Graphs

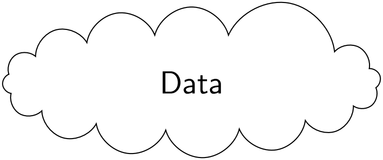
Daryl DeFord

MIT/Tufts
Metric Geometry and Gerrymandering Group

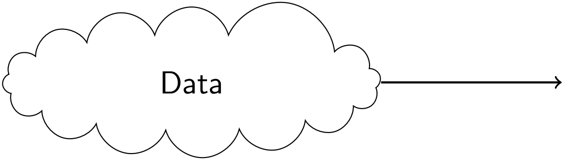
Math Department Colloquium
Tufts University
November 2, 2018

Outline

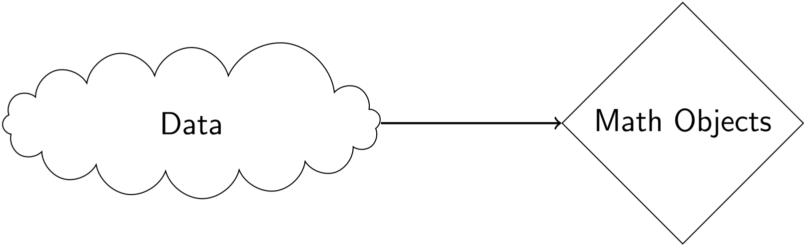
Philosophy



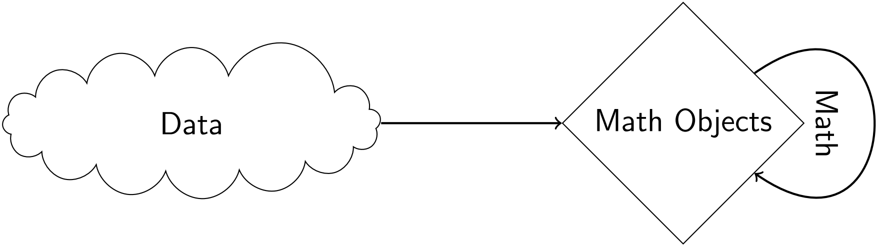
Philosophy



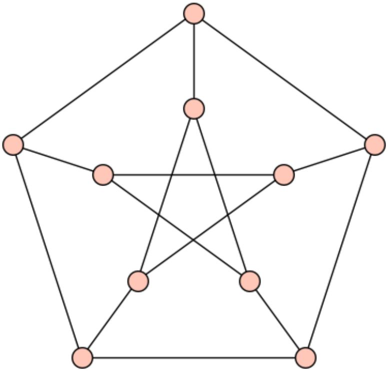
Philosophy



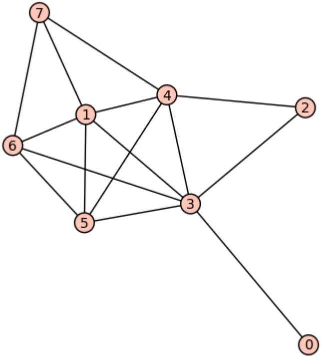
Philosophy



Graphs $(G = (V, E))$

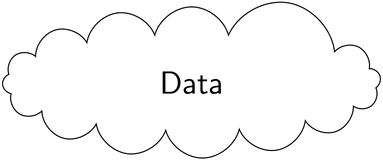


Adjacency Matrix

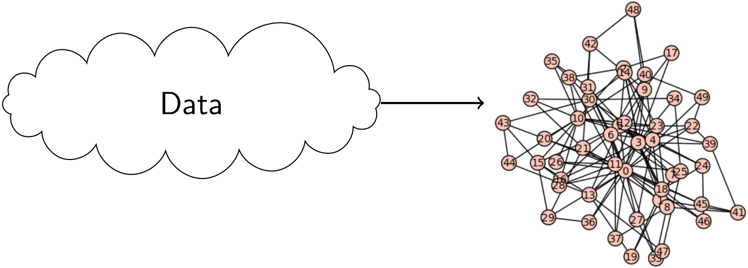


$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Networks



Networks



What is a multiplex?

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Definition

A *multiplex* is a collection of graphs all defined on the same node set.

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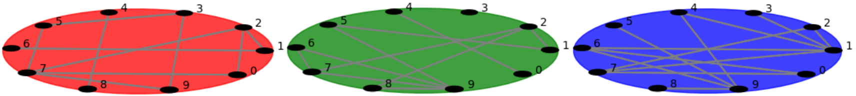
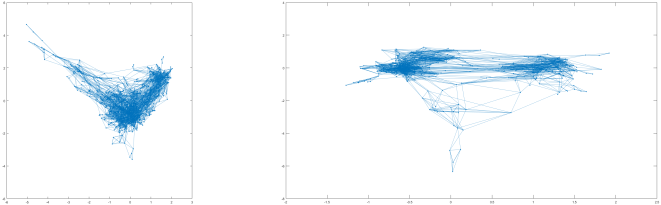


(a) Family (b) Colleagues (c) Facebook

Karnataka Village Data[1]



(a) Village 5

(b) Village 61

Figure: Karnataka Villages

[1] A. Banerjee, A.G. Chandrasekhar, E. Duflo, and M.O. Jackson, The Diffusion of Microfinance. Science, (2013).

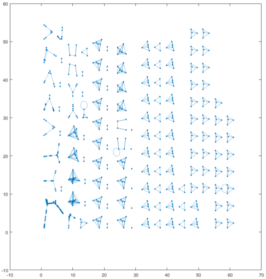
Village Layers

Layer Description	Village 5 Density	Village 5 Comp.	Village 5 Giant %	Village 61 Density	Village 61 Comp.	Village 61 Giant %
Borrow Money	.0082	26	.8354	.0108	15	.9188
Give Advice	.0077	49	.5892	.0098	34	.7377
Help Make Decisions	.0076	61	.1277	.0100	24	.8562
Borrow Kerosene or Rice	.0085	21	.8338	.0113	14	.9171
Lend Kerosene or Rice	.0086	22	.8308	.0113	14	.9255
Lend Money	.0081	14	.7908	.0107	17	.9036
Medical Advice	.0075	84	.2938	.0106	14	.9306
Friends	.0089	15	.9277	.0105	22	.8714
Relatives	.0085	29	.7231	.0105	26	.5448
Attend Temple With	.0073	117	.0462	.0089	108	.0372
Visit Their Home	.0087	15	.9185	.0116	11	.9475
Visit Your Home	.0088	16	.9108	.0117	11	.9492
Aggregate	.0121	3	.9862	.0155	8	.9679

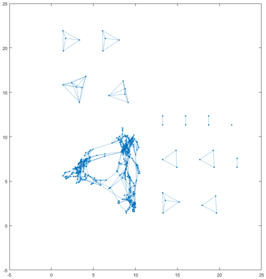
Table: Layer information for two of the Karnataka Villages[1].

[1] D. DeFord and S. Pauls, A new framework for dynamical models on multiplex networks, Journal of Complex Networks, (2018).

Medical Advice

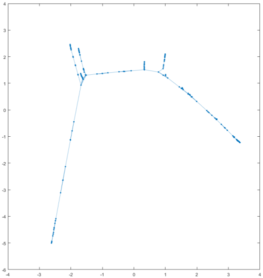


(a) Village 5

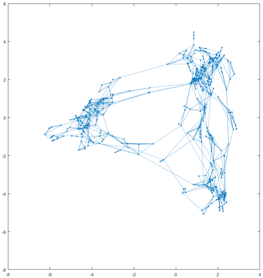


(b) Village 61

Medical Advice



(a) Village 5



(b) Village 61

World Trade Web[1]

Figure: World trade networks

[1] R. Feenstra, R. Lipsey, H. Deng, A.C. Ma, and H. Mo. World Trade Flows: 1962-2000. NBER Working Paper 11040, (2005).

WTW Layers

Layer	Description	Volume	% Total	Transitivity
0	Food and live animals	291554437	5.1	.82
1	Beverages and tobacco	48046852	0.9	.67
2	Crude materials	188946835	3.3	.79
3	Mineral fuels	565811660	10.0	.62
4	Animal and vegetable oils	14578671	0.3	.64
5	Chemicals	535703156	9.5	.83
6	Manufactured Goods	790582194	13.9	.87
7	Machinery	2387828874	42.1	.85
8	Miscellaneous manufacturing	736642890	13.0	.83
9	Other commodities	107685024	1.9	.56
All	Aggregate Trade	5667380593	100	.93

Table: Layer information for the 2000 World Trade Web[1].

[1] D. DeFord and S. Pauls, A new framework for dynamical models on multiplex networks, Journal of Complex Networks, (2018).

Disjoint Layers

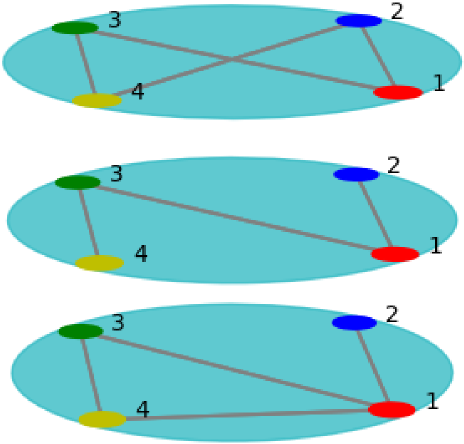
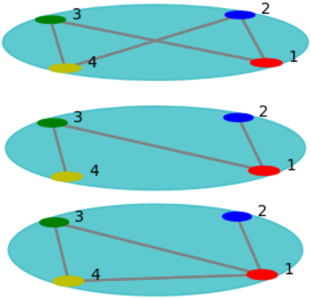
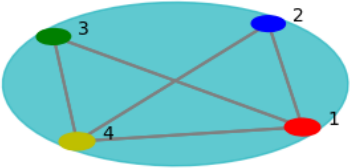


Figure: Disjoint Layers

Aggregate Representations

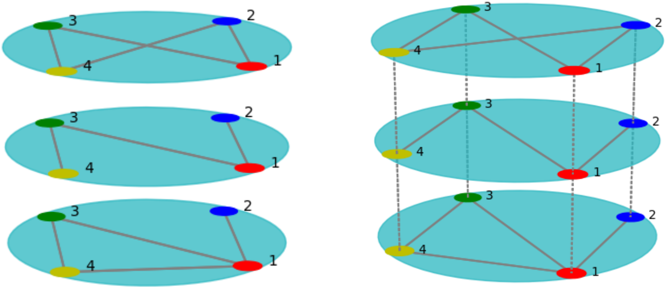


(a) Disjoint Layers



(b) Aggregate

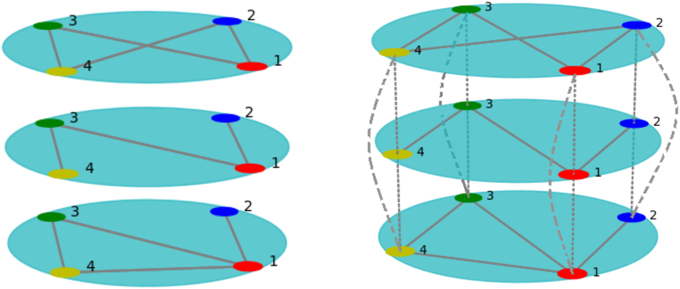
Temporal Multiplex



(a) Disjoint Layers (b) Temporal

P. J. Mucha, T. Richardson, K. Macon, M. A. Porter and J. P. Onnela, Community structure in time-dependent, multiscale, and multiplex networks, Science, (2010).

Supra–Adjacency



(a) Disjoint Layers (b) Supra–Adjacency

M. Kivelä, A. Arenas, M. Barthelemy, James P. Gleeson, Y. Moreno, M. A. Porter; Multilayer networks, Journal of Complex Networks, (2014).

Why Supra–Adjacency?

- Dynamical Properties [1]
- Generalizing Network Measures [2]
- Perturbation theory [3]

[1] S. Gómez, A. Díaz-Guilera, J. Gómez-Gardeñes, C. J. Pérez-Vicente, Y. Moreno, and A. Arenas, *Diffusion dynamics on multiplex networks*, Physical Review Letters 110 (2013), 2, 028701.

[2] M. De Domenico, A. Solé-Ribalta, E. Cozzo, M. Kivelä, Y. Moreno, M. A. Porter, S. Gómez, and A. Arenas, *Mathematical formulation of multilayer networks*, Physical Review X 3 (2013), 4, 041022.

[3] D. Taylor, S. A. Myers, A. Clauset, M. A. Porter and P. J. Mucha, Eigenvector-based centrality measures for temporal networks, Multiscale Modeling and Simulation 15, 537-574 (2017)

Matched Product

Definition (Matched Product)

Let $G_1, G_2, \ldots, G_k$ be an ordered list of graphs, each with n nodes and a common labeling of the nodes and let C be a graph with k ordered nodes. The matched product $\boxed{C}\,(G_1, G_2, \ldots, G_k)$ is the graph with node set $\bigcup V_i$ and two nodes v_i^{α} and v_j^{β} in $\boxed{C}\,(G_1, G_2, \ldots, G_k)$ are connected if and only if either

1. $c_\alpha \sim c_\beta$ and $i = j$
2. $\alpha = \beta$ and $v_i^{\alpha} \sim v_j^{\alpha}$

where c_α and c_β are nodes in C and v_i^{α} represents the copy of node i in G_α.

Example: Grid Graphs

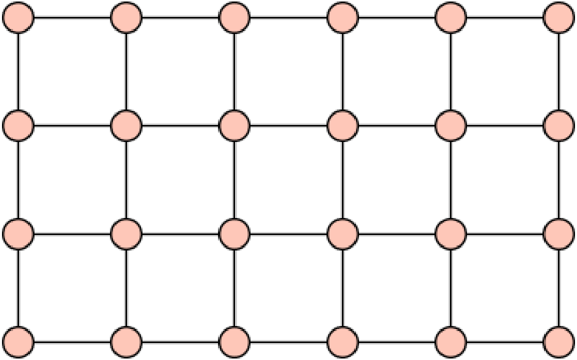


Figure: $\boxed{P_3}\,(P_6, P_6, P_6, P_6)$

Example: Hypercubes

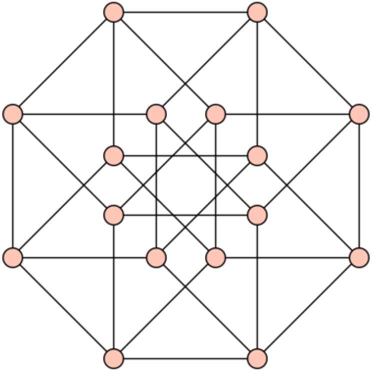


Figure:
$$\boxed{P_2}\left(\boxed{P_2}\left(\boxed{P_2}(P_2,P_2),\ \boxed{P_2}(P_2,P_2)\right),\ \boxed{P_2}\left(\boxed{P_2}(P_2,P_2),\ \boxed{P_2}(P_2,P_2)\right)\right)$$

Relationship to Other Graph Products

Theorem (D. 2017)

There are labelings of the graphs below such that the following hold:

1. *The cartesian product of G and H can be represented by* $\boxed{H}\ (G, G, \ldots, G)$
2. *The rooted product of G and H can be represented by* $\boxed{H}\ (G, E_n, E_n, \ldots, E_n)$
3. *The hierarchical product of G and H with subset $\{a_i\} \subset H$ can be represented by* $\boxed{H}\ (G_1, G_2, \ldots, G_k)$ *where*
$$G_i = \begin{cases} G & \text{if } i \in \{a_i\} \\ E_n & \text{otherwise} \end{cases}.$$

Product Proofs

Proof.

1. Two vertices (u, v) and (w, x) in the Cartesian product are connected if and only if $u \sim w$ or $v \sim x$. This is $\boxed{H}\ (G, G, \ldots, G)$: we note that the Cartesian condition is equivalent to construction $|G|$ copies of H and connecting all copies of a given label according to the topology of G.
2. The *rooted product* is constructed by fixing a root node in H and considering $|G|$ copies of H with the roots connected according to the topology of G. In $\boxed{H}\ (G, E_n, E_n, \ldots, E_n)$ we have a single copy of G and then n copies of H joined at the root by the topology of G exactly as desired. In general, filling the list $(G_1, G_2, \ldots, G_k)$ with copies of E_n simply gives that many disjoint copies of the graph H.

$\square$

Product Proof Continued

Proof.

3 The *hierarchical product* is a more recent invention, introduced in [1] and studied as a network model in [2]. Here the product is taken with respect to two graphs G and H, as well as a subset of the nodes of H. The construction begins with the rooted product of G and H but the copies of H that are associated to the subset are connected with the topology of G, instead of remaining empty. This is exactly the construction stated in the theorem in terms of the matched product.

$\square$

[1] L. Barrièrea, C. Dalfóa, M. A. Fiola, M. Mitjanab, The generalized hierarchical product of graphs, Discrete Mathematics, (2009).

[2] P. S. Skardal and K. Wash, Spectral properties of the hierarchical product of graphs, Physical Review E, (2016).

Cycle Properties

Proposition

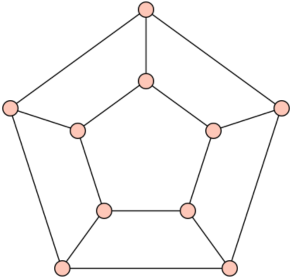
1. *Let $G_1, G_2, \ldots, G_k$ and C be Eulerian graphs then any labeling of $\boxed{C}\ (G_1, G_2, \ldots, G_k)$ is Eulerian.*
2. *Let $(G_1, G_2, \ldots, G_k)$ and C have Hamiltonian cycles. Then $\boxed{C}\ (G_1, G_2, \ldots, G_k)$ is Hamiltonian.*

Proof.

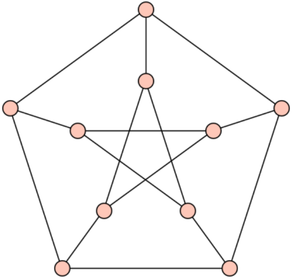
1. A graph is Eulerian if each vertex has even degree. Since the G_i and C are Eulerian each vertex in the product has even degree.
2. Label the nodes in $(G_1, G_2, \ldots, G_k)$ arbitrarily and let $(a_1, a_2, \ldots, a_k)$ be a Hamiltonian cycle in C. Starting at an arbitrary vertex in G_{a_1} traverse the cycle in G_{a_1} and then travel along the edge to G_{a_2}. The hypotheses guarantee that we can continue to traverse each layer, ending at a copy of the original node on G_{a_k}.

□

Labeling Matters!



(a) Cylinder Graph



(b) Petersen Graph

Figure: Two labelings of $\boxed{P_2}(C_5, C_5)$

Planarity Result

Proposition (Planarity)

Let G and H be connected graphs on n nodes. There exists a labeling so that of $\boxed{P_2}$ (G, H) *is planar if and only if G and H are outerplanar.*

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Let G and H be connected graphs on n nodes. There exists a labeling so that of $\boxed{P_2}\,(G,H)$ is planar if and only if G and H are outerplanar.

Lemma

Let G and H be connected, planar graphs. If $\boxed{P_2}\,(G,H)$ is planar then all vertices of G must lie within a single face of H and vice versa.

Proof of Lemma.

In order to obtain a contradiction, assume that two vertices u and v of G lie in different faces of H. Since G is connected there is a path connecting u and v but this path must pass through an edge of H which contradicts the assumption that $\boxed{P_2}\,(G,H)$ is planar. □

Planarity Proof Continued

Proof.

By the Lemma, if $\boxed{P_2}$ (G, H) is planar, each graph must have an embedding where each vertex lies on a single face, which is the definition of **outerplanar**[1].

The argument above shows that it is necessary for the graphs to be outerplanar, so it suffices to show that a compatible labeling exists. Since G and H are outerplanar, their vertices may be arranged at the vertices of a regular polygon in the plane, say with G centered at $(-1, 0)$ and H centered at $(1, 0)$ both inscribed in circles of radius $\frac{1}{2}$. To construct a planar labeling we choose a vertex arbitrarily from each graph and rotate the embeddings so that these are the two closest vertices. Pair these vertices and proceed along the ordering given by the polygon clockwise along the H and counterclockwise along G. □

[1] G. Chartrand, and F. Harary, Planar permutation graphs , Annales de l'Institut Henri Poincaré B, (1967).

Planar Path Labelings

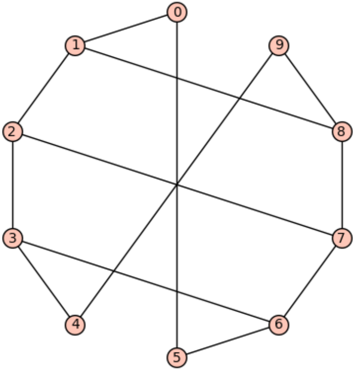

Figure: A labeling of $\boxed{P_2}\,(P_5, P_5)$ that is not planar.

Permutations of P_n

Theorem (D. 2018)

Let $\pi \in S_n$. Then $\boxed{P_2}\,(P_n, P_n)$ with labelings $(1, 2, 3, \ldots, n)$ and $(\pi(1), \pi(2), \pi(3), \ldots, \pi(n))$ is planar if and only if π is a square permutation. There are $2(n+2)4^{n-2} - 4(2n-5)\binom{2n-6}{n-3}$ such permutations.

Proof sketch.

A permutation is square if its consecutive–minima polygon has at most 4 sides. If π is square construct directly from diagram. If π is not square, there exists a vertex $2 < k < n-1$ such that contracting the edges between $1, \ldots, k-1$ and $k+1, \ldots, n$ is isomorphic to $K_{3,3}$. □

Permutation Examples

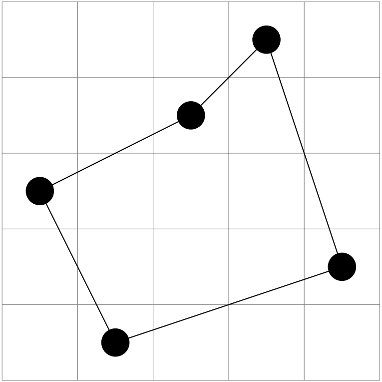

Figure: (3,1,4,5,2)

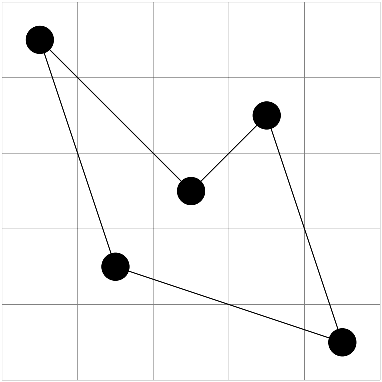

Figure: (5,2,3,4,1)

Original Problem (Honsberger)

Question

A classroom has 5 rows of 5 desks per row. The teacher requires that each pupil to change his seat by going either to the seat in front, the one behind, the one to his left, or the one to his right (of course not all these options are possible to all students). In how many ways can the students rearrange themselves?

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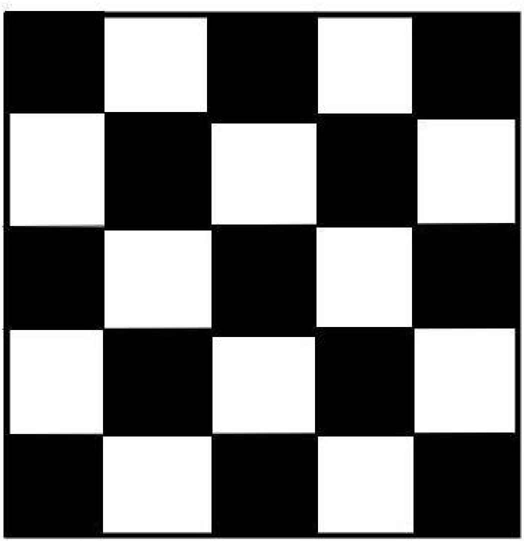
Answer

Zero.

Original Problem Solution



More interesting problem

Question

A classroom has 5 rows of 5 desks per row. The teacher **allows** *each pupil to change his seat by going either to the seat in front, the one behind, the one to his left, or the one to his right* **or to remain in place***. In how many ways can the students rearrange themselves?*

More interesting problem

Question

A classroom has 5 rows of 5 desks per row. The teacher **allows** *each pupil to change his seat by going either to the seat in front, the one behind, the one to his left, or the one to his right* **or to remain in place***. In how many ways can the students rearrange themselves?*

Answer

19,114,420

More interesting problem

Question

A classroom has 5 rows of 5 desks per row. The teacher **allows** *each pupil to change his seat by going either to the seat in front, the one behind, the one to his left, or the one to his right* **or to remain in place**. *In how many ways can the students rearrange themselves?*

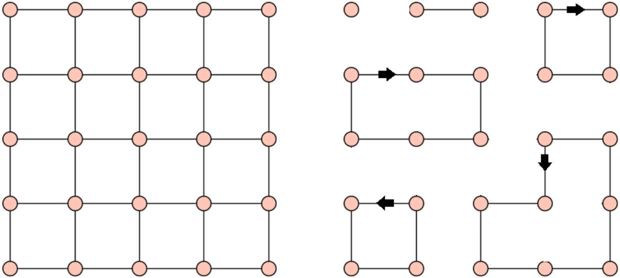
Answer

19,114,420

Definition (Graph Factorial)

The factorial of a graph G is the number of ways to decompose the vertices of G into a collection of disjoint cycles.

Example



(a) Original (b) Cycle Decomposition

Stirling Numbers of the first kind

Definition (Stirling Numbers)

The Stirling numbers of the first kind $\left[{n \atop k}\right]$ count the number of $\pi \in S_n$ composed of exactly k disjoint cycles. Since there are $n!$ elements of S_n we have:

$$n! = \sum_k \left[{n \atop k}\right].$$

Definition (Stirling Numbers for Graphs[1])

The Stirling numbers of the first kind $\left[{G \atop k}\right]$ count the number of ways to decompose the vertices of G into exactly k disjoint cycles and hence

$$G! = \sum_k \left[{G \atop k}\right].$$

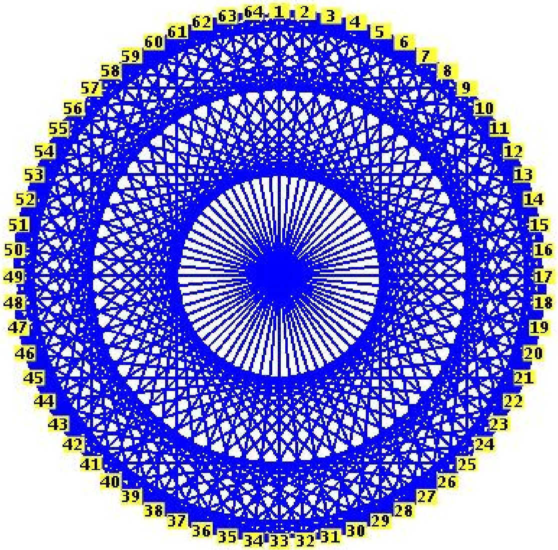
[1] A. Barghi, Stirling Numbers of the First Kind for Graphs, Australasian Journal of Combinatorics, (2018).

Chess Moves

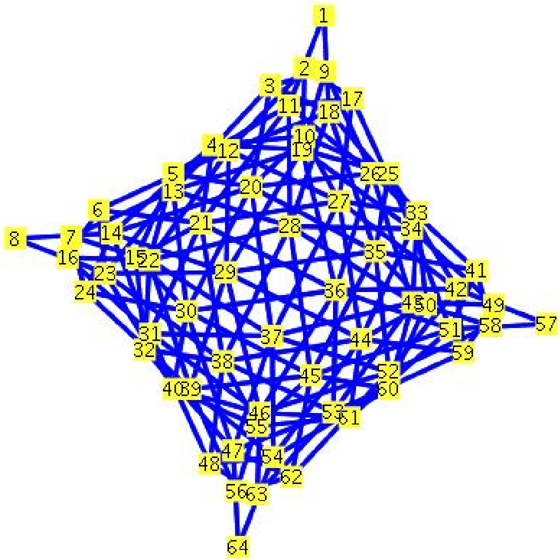
Question

A classroom has m rows of n desks per row. The teacher allows each pupil to change his seat by going either moving like **a given set of chess pieces** *or to remain in place. In how many ways can the students rearrange themselves?*

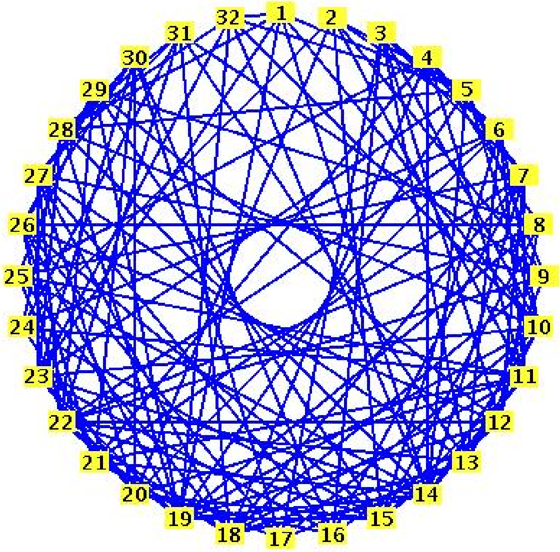
8×8 Rook Graph



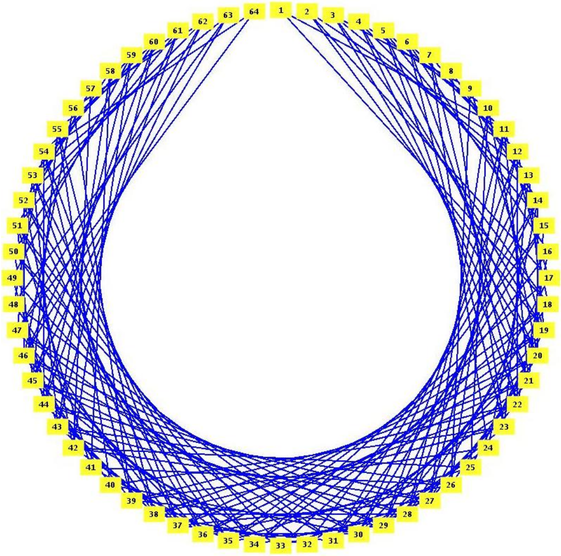
8×8 Knight Graph



8×8 Bishop Graph



Knight Rearrangements



Knight's Tour

- 8×8 Knight's Tour (Hamiltonian Cycles)
- 26,534,728,821,064 [1,2]

[1] M. Löbbing and I. Wegener, The Number of Knight's Tours Equals 33,439,123,484,294 — Counting with Binary Decision Diagrams, Electronic Journal of Combinatorics, (1996).

[2] B. McKay, Knight's Tours on an 8×8 Chessboard, Technical Report TR-CS-97-03, Australian National University, (1997).

Knight's Tour

- 8×8 Knight's Tour (Hamiltonian Cycles)
- 26,534,728,821,064 [1,2]
- 8×8 Knight's Graph Factorial
- **8,121,130,233,753,702,400**

[1] M. Löbbing and I. Wegener, The Number of Knight's Tours Equals 33,439,123,484,294 — Counting with Binary Decision Diagrams, Electronic Journal of Combinatorics, (1996).

[2] B. McKay, Knight's Tours on an 8×8 Chessboard, Technical Report TR-CS-97-03, Australian National University, (1997).

Matrix Permanents

Definition

The permanent of an $n \times n$ matrix, M, is defined by:

$$per(M) = \sum_{\pi \in S_n} \prod_{i=1}^{n} M_{i,\pi(i)}.$$

When $M = A$ is an adjacency matrix, this is exactly the number of cycle covers of the graph, that is:

$$per(A + I_n) = G!.$$

When $M = B$ is a bi–adjacency matrix, this is exactly the number of perfect matchings in the bipartite graph.

Convertible Graphs

Definition

A graph is called convertible if we can scale the entries of the adjacency matrix by units to obtain a new matrix B such that:

$$per(A) = det(B).$$

- Posed by Pólya
- Planar bi–adjacency case by FKT
- Characterization as Pfaffian orientable bi–adjacency graphs [1]
- Polynomial Time Algorithm [2]

[1] C. H. Little, A characterization of convertible (0, 1)-matrices, Journal of Combinatorial Theory, Series B, (1975).

[2] N. Robertson, P.D. Seymour, and R. Thomas, Permanents, Pfaffian orientations, and even directed circuits, Annals of Mathematics, (1999).

Example: Hosoya Index of Trees

Theorem (D. 2013)

Let T be a tree with adjacency matrix $A(T)$. Then the Hosoya index of T is equal to

$$det(iA(T) + I_n).$$

If G has exactly one k cycle C_k, then:

$$G! = det(iA(G) + I_n) + 2(-i)^k(det(iA(G \setminus C) + I_{n-k}).$$

Proof.

Since T is a tree there is a bijection between cycle covers and matchings.

$$\begin{aligned} det(iA + I_n) &= \sum_{\pi \in S_n} sgn(\pi) \prod_i (iA + I_n)_{i,\pi(i)} \\ &= \sum_{\pi \in S_n} sgn(\pi)^2 \prod_i (A + I_n)_{i,\pi(i)} \\ &= \sum_{\pi \in S_n} \prod_i (A + I_n)_{i,\pi(i)} \\ &= per(A + I_n) = T! \end{aligned}$$

□

Chessboard Theorem

Theorem (D. 2014)

Let M be a set of adjacency moves on a chessboard with bounded horizontal displacement and let $\{G_i\}$ be a sequence of graphs with V_i representing the squares of an $m \times i$ grid and E_i defined by the moves in M. Then, the sequence $G_1!, G_2!, G_3!, \ldots$ satisfies a linear, homogeneous, constant–coefficient recurrence relation.

Example

Let M be the bishop move and $m = 2$. Then, $G_n! = f_n^2$

Graph Family Factorials

G_n	$G_n!$
P_n	f_n
C_n	$f_n + f_{n-2} + 2$
K_n	$n!$
$K_{m,n}$	$\sum_{i=0}^{m} (m)_i (n)_i$
$Star_n$	$n+1$
$Wheel_n$	$nf_{n+2} + f_n + f_{n-2} - 2n + 2$
$Dutch\ Windmill_n^m$	$(f_{n-1})^m + 2(f_{n-2}+1)(f_{n-1})^{m-1}$
$Flower\ Graph_n^k$	$2 + \ell_{(k-2)n} + nf_{(k-2)n-1} + 2nf_{(n-2)k-(n-1)} +$ $2n \sum_{i=1}^{k-2} f_{(k-2_{(}n-i-1)-1}$

Comb Graph Factorials

G_n	$G_n!$
$\boxed{P_2}\ (E_n, E_n)$	2^n
$\boxed{P_2}\ (P_n, E_n)$	L_n
$\boxed{P_2}\ (S_n, E_n)$	$2^{n+1} + n2^n$
$\boxed{P_2}\ (C_n, E_n)$	$2L_{n-1} + 2L_{n-2} + 4$
$\boxed{P_2}\ (K_n, E_n)$	$\sum_\ell \binom{n}{\ell}(n-\ell)!$

The Pell numbers, L_n, are defined by $L_0 = 1$, $L_1 = 2$, and $L_n = 2L_{n-1} + L_{n-2}$.

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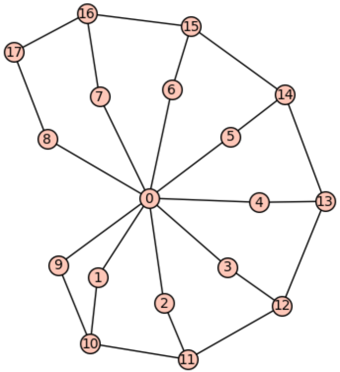
$\boxed{P_2}\ (C_n, E_n)$	$2L_{n-1} + 2L_{n-2} + 4$
$\boxed{P_2}\ (K_n, E_n)$	$\sum_\ell \binom{n}{\ell}(n-\ell)!$
$\boxed{P_2}\ (C_n, C_n)$	$6 + 4(-1)^n + \left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n + \left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n$

The Pell numbers, L_n, are defined by $L_0 = 1$, $L_1 = 2$, and $L_n = 2L_{n-1} + L_{n-2}$.

$\boxed{P_2}$ (S_9, P_9)



$\boxed{P_2}\ (S_n, P_n)$

Example

$$\begin{aligned}\boxed{P_2}\ (S_n, P_n)! =& 2L_{n+1} + (L_{j-1} + L_{j-2})L_{n-j-1} \\ &+ \sum_{j=1}^{n} [L_{j-1} + 2L_{j-2} + L_{n-3}]L_{n-j} \\ &+ 2\left(\sum_{j=1}^{n} L_{n-j} + \sum_{j=1}^{n-1} [L_{j-1} + L_{j-2}] \sum_{m=j+1}^{n} L_{n-m}\right)\end{aligned}$$

Extensions

- Expected enumeration over relabelings:
  - Planar
  - Factorial
  - Chromatic Number
  - etc.
- Products that are plan for all labelings

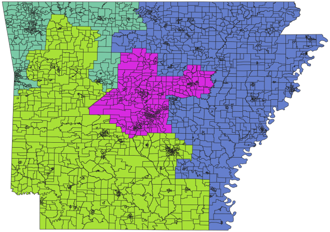
$$\boxed{P_2}\,(S_n, P_n) \text{ and } \boxed{P_2}\,(S_n, S_n)$$

- Products that are isomorphic for all labelings

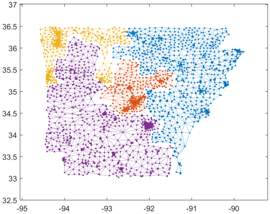
$$\boxed{P_2}\,(K_n, G) \text{ and } \boxed{P_2}\,(S_n, C_n)$$

- Products that are never isomorphic for all labelings

Political Partitioning



(a) Geography



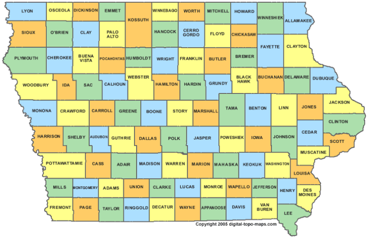
(b) Dual Graph

Districting Plans

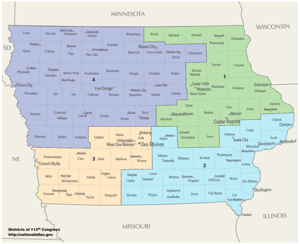
We want to partition a given geography (graph), at a given scale, into k pieces, satisfying some constraints:

- Contiguity
- Population Balance
- Compactness
- Communities of Interest
- Municipal Boundaries
- Competitiveness/Responsiveness
- Incumbency Protection
- ...

Nesting Rules



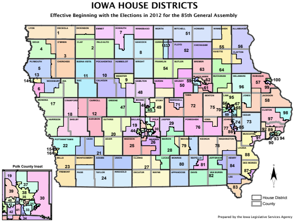
(a) Iowa Counties



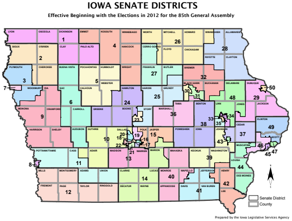
(b) Congressional Districts

Figure: Still quintillions of possibilities...

Perfect Matchings



(a) House Districts



(b) Senate Districts

Figure: 100 House districts paired to make 50 Senate districts.

Matching Advantages

Approximately 8 states require this type of matching and two states (OH and WI) have a three house districts to one senate district matching.

- Matchings are contiguous
- Population balance is automatic
- Compactness is less relevant
- ...

Even better, for graphs of this size and type we can construct all of the possible matchings. This lets us evaluate whether or not a given matching is a partisan outlier and make statements about the actual space of possibilities.

(VRDI) Alaska

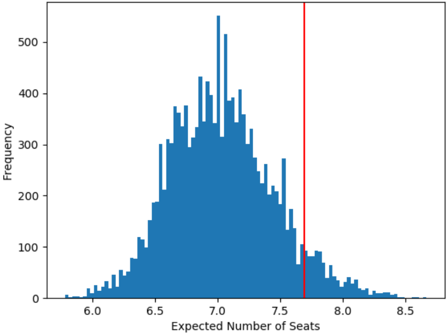


Figure: Histogram of expected Democratic senate seats across all matchings.

Analysis by Caldera, Duchin, Elhai, Gutekunst, Harris, Kelling, Khan, and Nix

Extensions

- How bad can it get in practice?
- Can we efficiently generate the set of "triple pairings" on graphs of this size?
- How does perturbing the smaller districts change the matching properties?
- If we instead started with Senate districts and bipartition them into house districts, what can we say about the distributions?
- Is there a general, multiscale approach that could be applied to sample plans, even when the state doesn't require pairing?

The end!

Thanks!