# Èdouard Lucas:

*The theory of recurrent sequences is an inexhaustible mine which contains all the properties of numbers; by calculating the successive terms of such sequences, decomposing them into their prime factors and seeking out by experimentation the laws of appearance and reproduction of the prime numbers, one can advance in a systematic manner the study of the properties of numbers and their application to all branches of mathematics.*

# Matched Products and Stirling Numbers of Graphs

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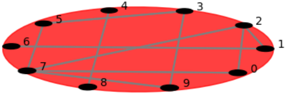
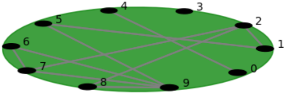
# Multilayer Networks

**Definition**

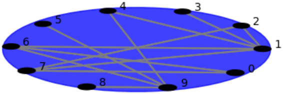
A *multilayer network* is a collection of graphs all defined on the same node set. Formally, $M = (V, (E_1, E_2, \dots, E_k))$ where $G_i = (V, E_i)$ is a graph for all $i$.



(a) Family



(b) Colleagues



(c) Facebook

# Disjoint Layers

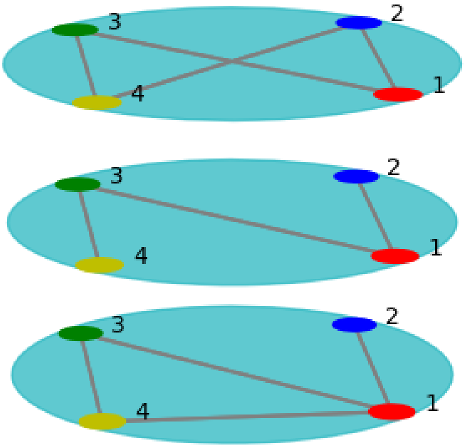
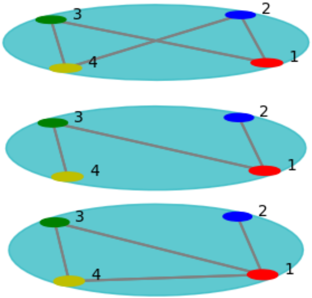
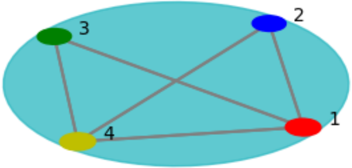



Figure: Disjoint Layers

# Aggregate Representations

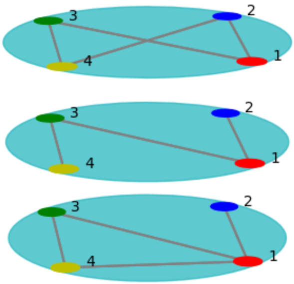



(a) Disjoint Layers

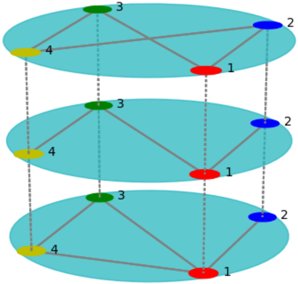



(b) Aggregate

# Temporal Data



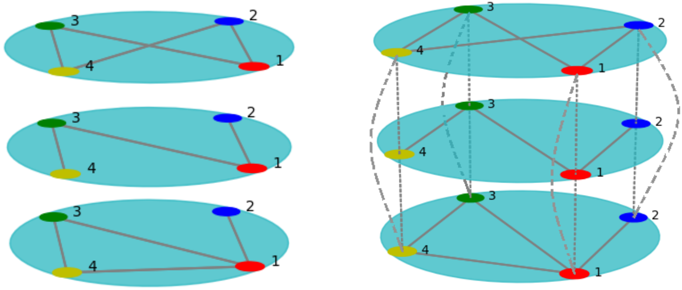

(a) Disjoint Layers




(b) Temporal

P. J. Mucha, T. Richardson, K. Macon, M. A. Porter and J. P. Onnela, Community structure in time-dependent, multiscale, and multiplex networks, Science, (2010).

# Supra–Adjacency




(a) Disjoint Layers (b) Supra–Adjacency

M. Kivelä, A. Arenas, M. Barthelemy, James P. Gleeson, Y. Moreno, M. A. Porter; Multilayer networks, Journal of Complex Networks, (2014).

# Why Supra–Adjacency?

- Dynamical Properties [1]
- Generalizing Network Measures [2]
- Perturbation Theory [3]

[1] S. Gómez, A. Díaz-Guilera, J. Gómez-Gardeñes, C. J. Pérez-Vicente, Y. Moreno, and A. Arenas, *Diffusion dynamics on multiplex networks*, Physical Review Letters 110 (2013), 2, 028701.

[2] M. De Domenico, A. Solé-Ribalta, E. Cozzo, M. Kivelä, Y. Moreno, M. A. Porter, S. Gómez, and A. Arenas, *Mathematical formulation of multilayer networks*, Physical Review X 3 (2013), 4, 041022.

[3] D. Taylor, S. A. Myers, A. Clauset, M. A. Porter and P. J. Mucha, Eigenvector-based centrality measures for temporal networks, Multiscale Modeling and Simulation 15, 537-574 (2017)

# Matched Product

**Definition (Matched Product)**

Let $G_1, G_2, \ldots, G_k$ be an ordered list of graphs, each with $n$ nodes and a common labeling of the nodes and let $C$ be a graph with $k$ ordered nodes. The matched product $\boxed{C}\,(G_1, G_2, \ldots, G_k)$ is the graph with node set $\bigcup V_i$ and two nodes $v_i^\alpha$ and $v_j^\beta$ in $\boxed{C}\,(G_1, G_2, \ldots, G_k)$ are connected if and only if either

1. $c_\alpha \sim c_\beta$ and $i = j$
2. $\alpha = \beta$ and $v_i^\alpha \sim v_j^\alpha$

where $c_\alpha$ and $c_\beta$ are nodes in $C$ and $v_i^\alpha$ represents the copy of node $i$ in $G_\alpha$.

# Example: Petersen Graph

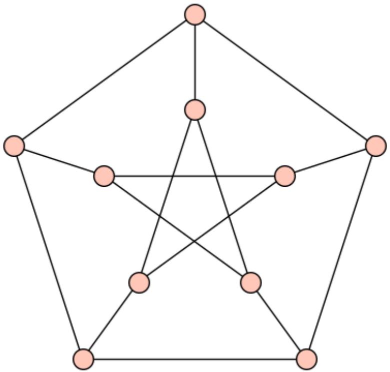


Figure: $\boxed{P_2}\,(C_5, C_5)$

# Example: Hypercubes

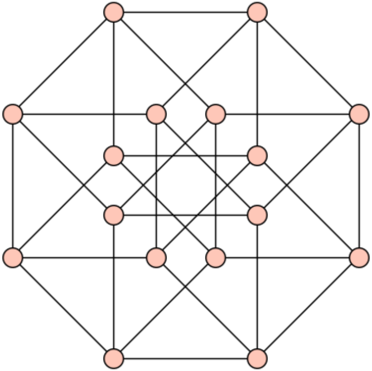


Figure:
$$\boxed{P_2}\left(\boxed{P_2}\left(\boxed{P_2}(P_2,P_2),\ \boxed{P_2}(P_2,P_2)\right),\ \boxed{P_2}\left(\boxed{P_2}(P_2,P_2),\ \boxed{P_2}(P_2,P_2)\right)\right)$$

# Relationship to Other Graph Products

**Theorem (D. 2017)**

*There are labelings of the graphs below such that the following hold:*

1. *The cartesian product of $G$ and $H$ can be represented by* $\boxed{H}\,(G, G, \ldots, G)$
2. *The rooted product of $G$ and $H$ can be represented by* $\boxed{H}\,(G, E_n, E_n, \ldots, E_n)$
3. *The hierarchical product*[1,2] *of $G$ and $H$ with subset $\{a_i\} \subset H$ can be represented by* $\boxed{H}\,(G_1, G_2, \ldots, G_k)$ *where*

$$G_i = \begin{cases} G & \textit{if } i \in \{a_i\} \\ E_n & \textit{otherwise} \end{cases}.$$

[1] L. Barrièrea, C. Dalfóa, M. A. Fiola, M. Mitjanab, The generalized hierarchical product of graphs, Discrete Mathematics, (2009).

[2] P. S. Skardal and K. Wash, Spectral properties of the hierarchical product of graphs, Physical Review E, (2016).

# Cycle Properties

## Proposition

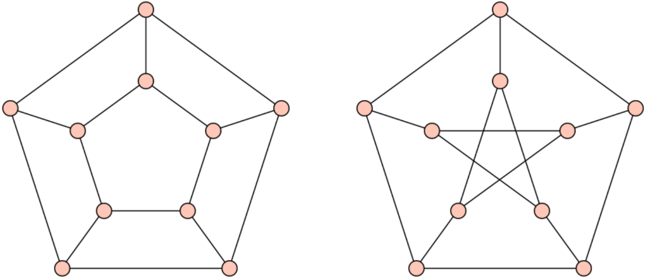
1. *Let $G_1, G_2, \dots, G_k$ and $C$ be Eulerian graphs then any labeling of* $\boxed{C}\,(G_1, G_2, \dots, G_k)$ *is Eulerian.*
2. *Let $(G_1, G_2, \dots, G_k)$ and $C$ have Hamiltonian cycles. Then* $\boxed{C}\,(G_1, G_2, \dots, G_k)$ *is Hamiltonian.*

## Proof.

1. A graph is Eulerian if each vertex has even degree. Since the $G_i$ and $C$ are Eulerian each vertex in the product has even degree.
2. Label the nodes in $(G_1, G_2, \dots, G_k)$ arbitrarily and let $(a_1, a_2, \dots, a_k)$ be a Hamiltonian cycle in $C$. Starting at an arbitrary vertex in $G_{a_1}$ traverse the cycle in $G_{a_1}$ and then travel along the edge to $G_{a_2}$. The hypotheses guarantee that we can continue to traverse each layer, ending at a copy of the original node on $G_{a_k}$.

□

# Labeling Matters!



(a) Cylinder Graph (b) Petersen Graph

Figure: Two labelings of $\boxed{P_2}\,(C_5, C_5)$

# Planarity Result

**Proposition (Planarity)**

*Let $G$ and $H$ be connected graphs on $n$ nodes. There exists a labeling so that of* $\boxed{P_2}$ $(G, H)$ *is planar if and only if $G$ and $H$ are outerplanar*[1].

[1] G. Chartrand, and F. Harary, Planar permutation graphs , Annales de l'Institut Henri Poincaré B, (1967).

# Planarity Result

**Proposition (Planarity)**

*Let $G$ and $H$ be connected graphs on $n$ nodes. There exists a labeling so that of $\boxed{P_2}$ $(G, H)$ is planar if and only if $G$ and $H$ are outerplanar*[1].

**Lemma**

*Let $G$ and $H$ be connected, planar graphs. If $\boxed{P_2}$ $(G, H)$ is planar then all vertices of $G$ must lie within a single face of $H$ and vice versa.*

[1] G. Chartrand, and F. Harary, Planar permutation graphs , Annales de l'Institut Henri Poincaré B, (1967).

# Planar Path Labelings

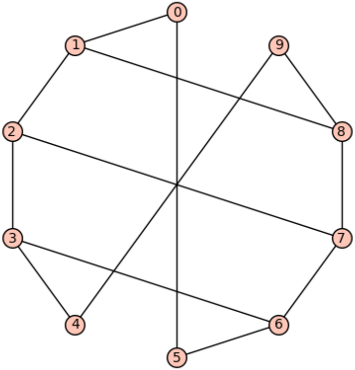



Figure: A labeling of $\boxed{P_2}\,(P_5, P_5)$ that is not planar.

# Permutations of $P_n$

## Theorem (D. 2018)

*Let $\pi \in S_n$. Then $\boxed{P_2}$ $(P_n, P_n)$ with labelings $(1, 2, 3, \ldots, n)$ and $(\pi(1), \pi(2), \pi(3), \ldots, \pi(n))$ is planar if and only if $\pi$ is a square permutation. There are $2(n+2)4^{n-2} - 4(2n-5)\binom{2n-6}{n-3}$ such permutations.*

## Proof sketch.

A permutation is square if its consecutive–minima polygon has at most 4 sides. If $\pi$ is square construct directly from diagram. If $\pi$ is not square, there exists a vertex $2 < k < n-1$ such that contracting the edges between $1, \ldots, k-1$ and $k+1, \ldots, n$ is isomorphic to $K_{3,3}$. □

# Permutation Examples

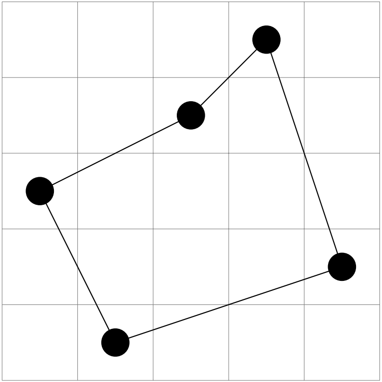


Figure: (3,1,4,5,2)

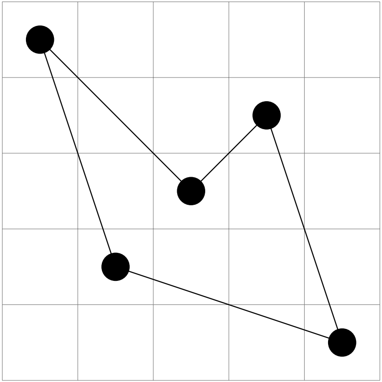


Figure: (5,2,3,4,1)

# Stirling Numbers of the first kind

**Definition (Graph Factorial)**

The factorial of a graph $G$ is the number of ways to decompose the vertices of $G$ into a collection of disjoint cycles.

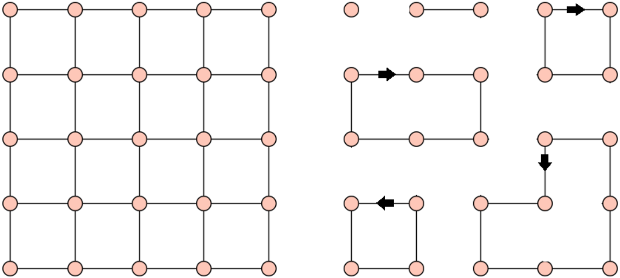
**Definition (Stirling Numbers for Graphs[1])**

The Stirling numbers of the first kind $\left[{G \atop k}\right]$ count the number of ways to decompose the vertices of $G$ into exactly $k$ disjoint cycles and hence

$$G! = \sum_k \left[{G \atop k}\right].$$

[1] A. Barghi, Stirling Numbers of the First Kind for Graphs, Australasian Journal of Combinatorics, (2018).

# Cycle Decomposition Example



(a) Original Graph

(b) Cycle Decomposition

# Comb Graph Factorials

| $G_n$ | $G_n!$ |
|---|---|
| $\boxed{P_2}\ (E_n, E_n)$ | $2^n$ |
| $\boxed{P_2}\ (P_n, E_n)$ | $L_n$ |
| $\boxed{P_2}\ (S_n, E_n)$ | $2^{n+1} + n2^n$ |
| $\boxed{P_2}\ (C_n, E_n)$ | $2L_{n-1} + 2L_{n-2} + 4$ |
| $\boxed{P_2}\ (K_n, E_n)$ | $\sum_\ell \binom{n}{\ell}(n-\ell)!$ |
| $\boxed{P_2}\ (C_n, C_n)$ | $6 + 4(-1)^n + \left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n + \left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n$ |

The Pell numbers, $L_n$, are defined by $L_0 = 1$, $L_1 = 2$, and $L_n = 2L_{n-1} + L_{n-2}$.

# $\boxed{P_2}\ (S_n, P_n)$

## Example

$$\begin{aligned}\boxed{P_2}\ (S_n, P_n)! =& 2L_{n+1} + (L_{j-1} + L_{j-2})L_{n-j-1} \\ &+ \sum_{j=1}^{n} [L_{j-1} + 2L_{j-2} + L_{n-3}]L_{n-j} \\ &+ 2\left(\sum_{j=1}^{n} L_{n-j} + \sum_{j=1}^{n-1} [L_{j-1} + L_{j-2}] \sum_{m=j+1}^{n} L_{n-m}\right)\end{aligned}$$

# Extensions

- Expected enumeration over relabelings:
  - Planarity
  - Graph Factorials
  - Chromatic Number
  - (insert your favorite graph metric here)
- Products that are planar for all labelings

$$\boxed{P_2}\,(S_n, P_n) \text{ and } \boxed{P_2}\,(S_n, S_n)$$

- Products that are isomorphic for all labelings

$$\boxed{P_2}\,(K_n, G) \text{ and } \boxed{P_2}\,(S_n, C_n)$$

- Products that are never isomorphic for all labelings

# The end

Thanks!