# Dynamics on Multiplex Networks

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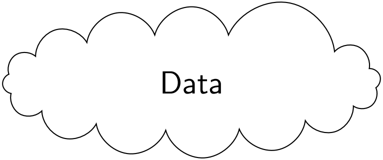
# Outline

# Philosophy



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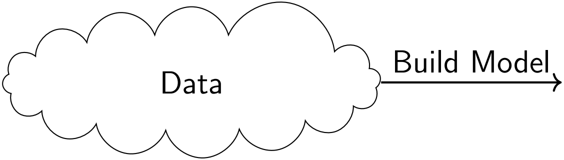

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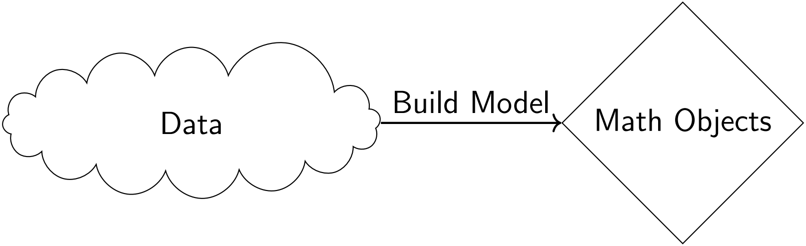

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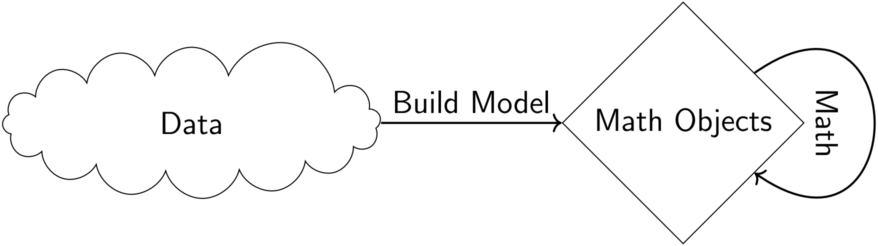

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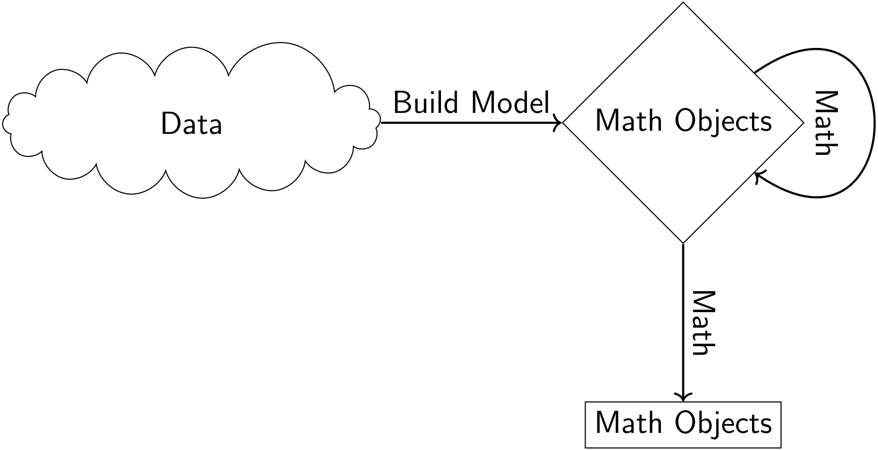
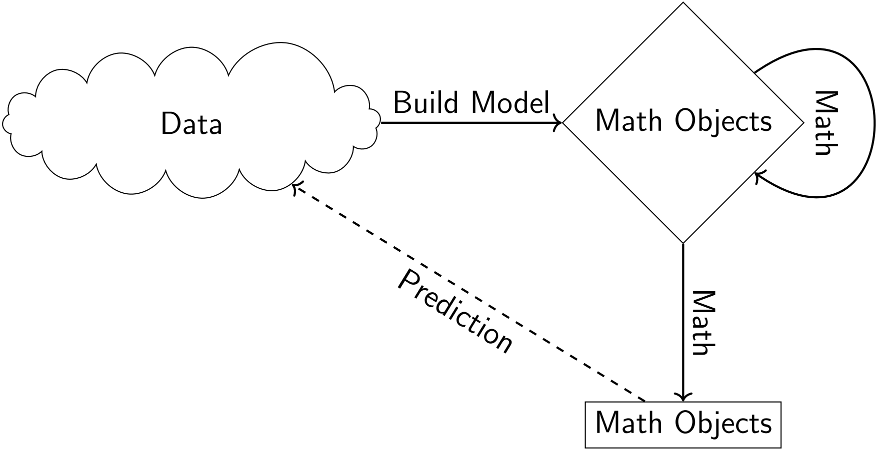
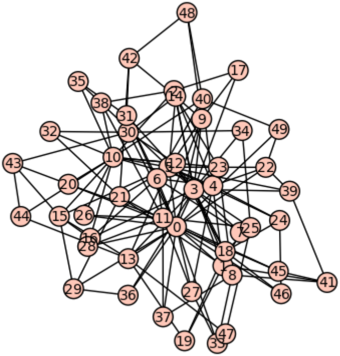


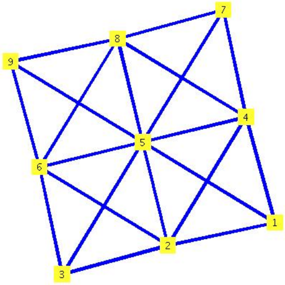
# Philosophy


Data
Build Model
Math Objects
Math
Math
Prediction
Math Objects

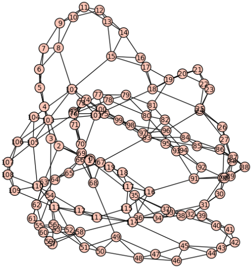
# Complex Networks



# Examples

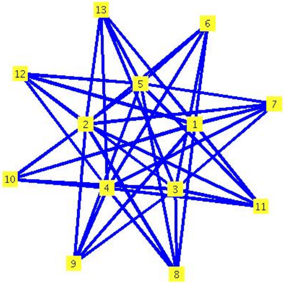



(a) Graph

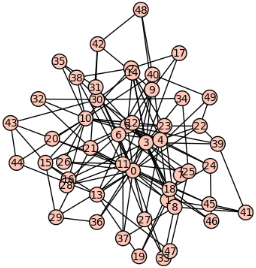


(b) Network

# Examples

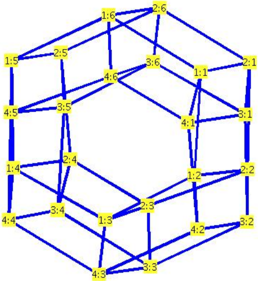



(a) Graph

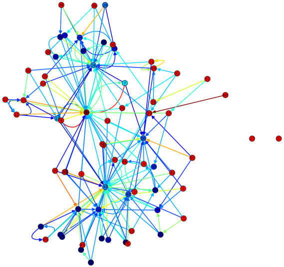


(b) Network

# Examples

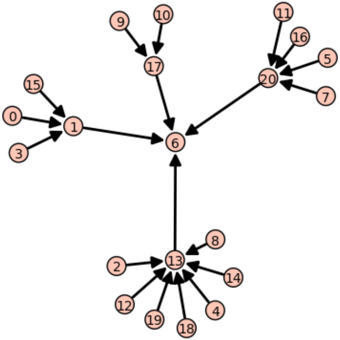



(a) Graph

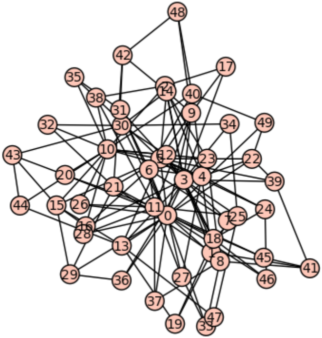


(b) Network

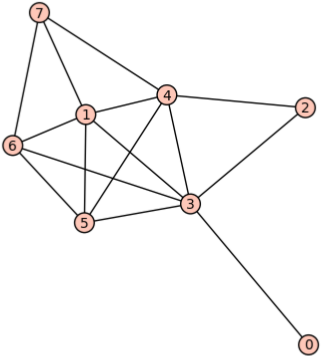
# Centrality



# Centrality

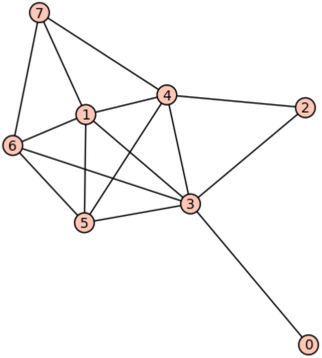

48
42
17
35
2
14
40
38
31
9
32
30
34
49
43
10
12
23
22
6
3
4
39
20
21
44
15
26
11
0
24
7
25
16
28
18
13
45
8
41
29
36
27
46
37
19
33
47

# Degree Matrix




$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

# Adjacency Matrix




$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

# Operators on Networks

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- Adjacency
  - Symmetric, binary
  - Eigenvector centrality

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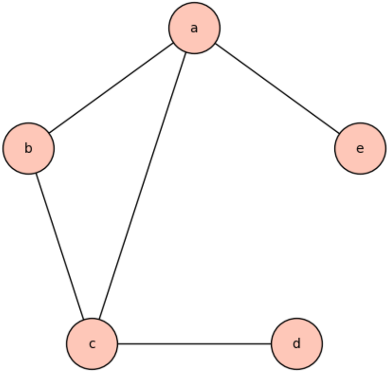
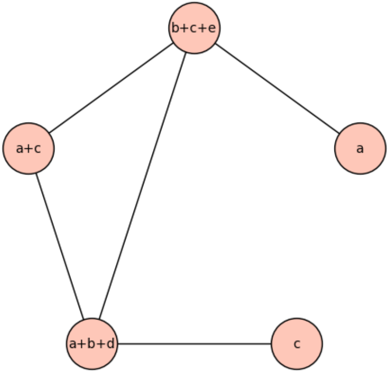
- Adjacency
  - Symmetric, binary
  - Eigenvector centrality
- Laplacian
  - Matrix: $L = D - A$
  - Positive semi–definite
  - Discretized version of Laplacian heat diffusion
- Random Walk
  - Matrix; $R = D^{-1}A$
  - Always stochastic, regular if $G$ is connected
  - Transition matrix of associated Markov process

# Adjacency Action

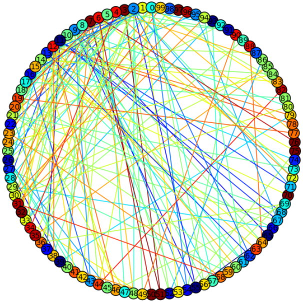


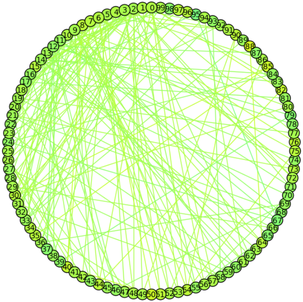

(c) Before



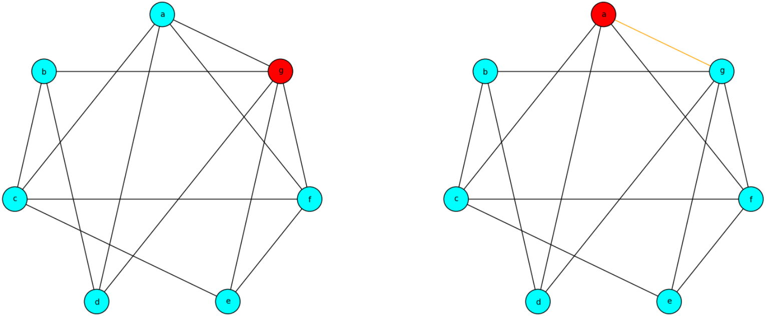

(d) After

# Diffusion Animation

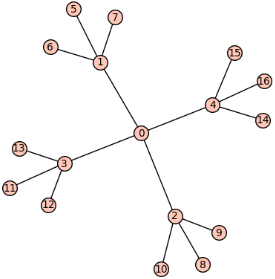




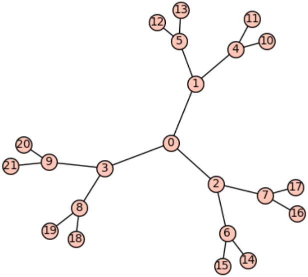
# Random Walk Animation



# Adjacency Spectra of Regular Trees




(c) $X_2^{(4)}$




(d) $X_3^{(3)}$

# Eigenvalues of (3)–trees

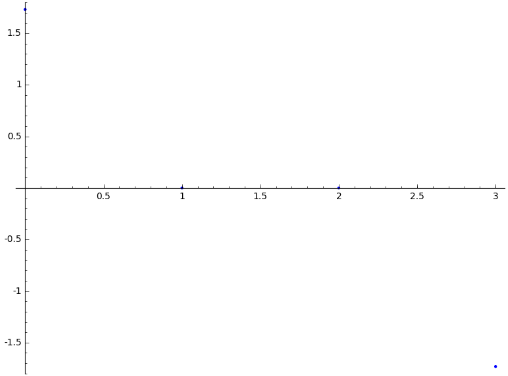



Figure: Eigenvalues of $X_1^{(3)}$

# Eigenvalues of (3)–trees

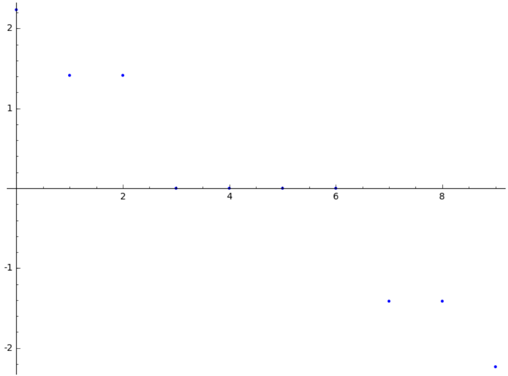



Figure: Eigenvalues of $X_2^{(3)}$

# Eigenvalues of (3)–trees

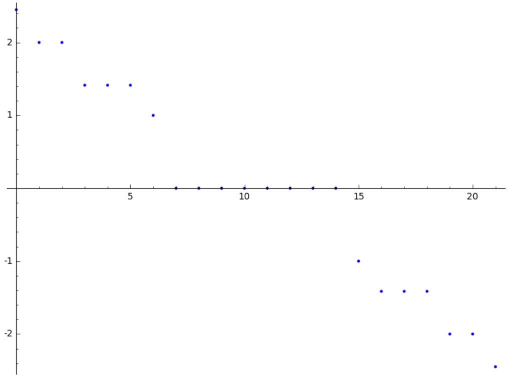



Figure: Eigenvalues of $X_3^{(3)}$

# Eigenvalues of (3)–trees

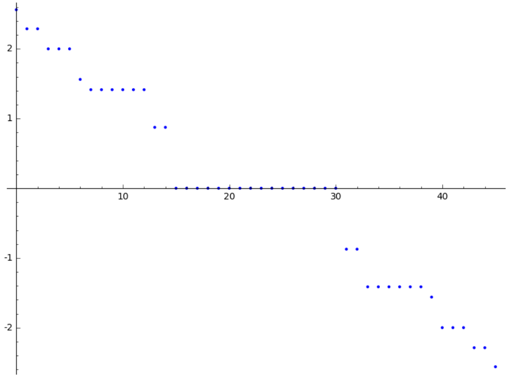



Figure: Eigenvalues of $X_4^{(3)}$

# Eigenvalues of (3)–trees

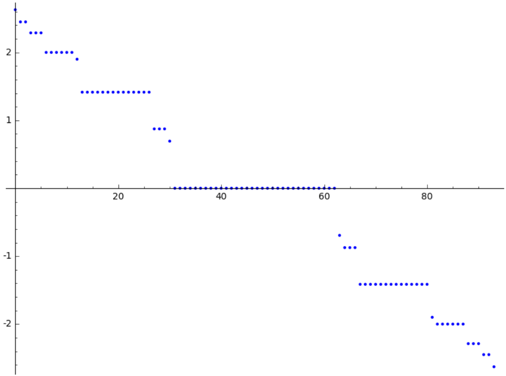



Figure: Eigenvalues of $X_5^{(3)}$

# Eigenvalues of (3)–trees

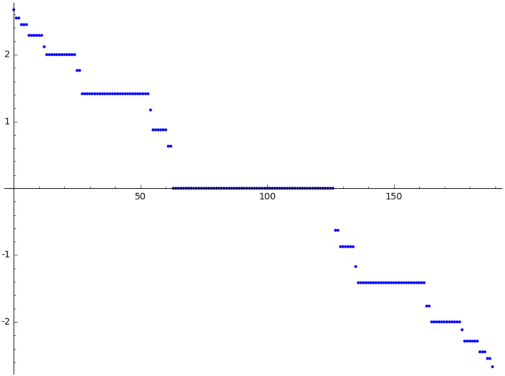



Figure: Eigenvalues of $X_6^{(3)}$

# Eigenvalues of (3)–trees

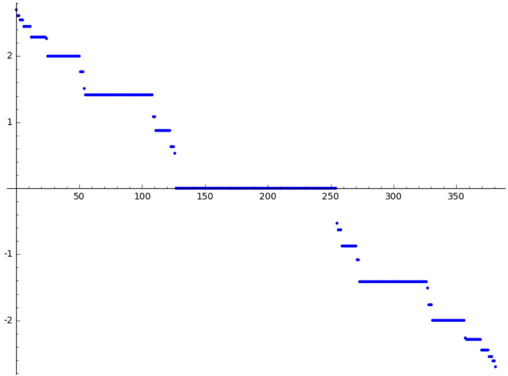



Figure: Eigenvalues of $X_7^{(3)}$

# Eigenvalues of (3)–trees

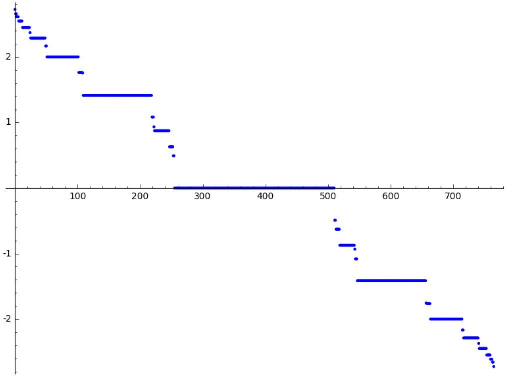



Figure: Eigenvalues of $X_8^{(3)}$

# Eigenvalues of Regular Trees

Define two families of polynomials:

$$P_n^k(x) = xP_{n-1}^k - (k-1)P_{n-2}^k$$

with initial conditions $P_0^k(x) = 0$, $P_1^k(x) = 1$, and $P_2^k(x) = x$ and

$$Q_n^k(x) = xP_n^k(x) - kP_{n-1}^k(x).$$

**Theorem**

*The roots of $P_s^k(x)$ for $1 \leq s \leq r$ and $Q_r^k(x)$ are precisely the eigenvalues of the finite $k$-ary tree $X_r^k$.*

# Enumerative Results

**Theorem**

*If $\lambda$ is a root $P_r^k$ and not a root of $P_m^k$ for any $m < r$ then asymptotically (as $r \longrightarrow \infty$), the proportion of eigenvalues of $X_k^r$ is* $\dfrac{(k-2)^2}{(k-1)^r - 1}$.

**Corollary**

$$\sum_{n=2}^{\infty} \frac{\varphi(n)(k-2)^2}{(k-1)^n - 1} = 1.$$

**Corollary ($k = 3$)**

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{2^n - 1} = 2.$$

# Tree Questions

## Question

*Given a graph $G$ and an associated adjacency eigenpair $(v, \lambda)$ does there exist a subgraph $H$ of $G$ so that $(v|_H, \lambda)$ is an eigenpair for $H$?*

## Question

*Is there a nice closed form for the endpoints of the Cantor–like set:*

$$\left\{ \sum_{n=1}^{\infty} \frac{(k-2)^2}{(k-1)^n - 1} \sum_{\substack{(\ell,n)=1 \\ \ell < \frac{an}{m}}} 1 : \begin{array}{c} m \in \mathbb{N} \\ (a,m) = 1 \end{array} \right\}$$

# Tree Questions

## Question

*Can we characterize the sequences of graphs $G_1, G_2, \ldots$ satisfying for all $\varepsilon > 0$ there exists a finite set $\Lambda \subset \mathbb{R}$ and a $N \in \mathbb{N}$ such that for all $n > N$:*

$$\frac{|\{\lambda \in \mathrm{spec}(G_n) : \lambda \notin \Lambda\}|}{|\mathrm{spec}(G_n)|} < \varepsilon.$$

## Question

*What similar results exist for regularly branching simplices?*

# What is a multiplex?

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**Definition**

A *multiplex* is a collection of graphs all defined on the same node set.

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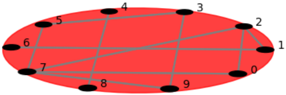
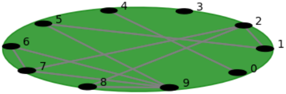
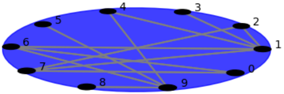



(a) Family




(b) Colleagues




(c) Facebook

# World Trade Web[1]

Figure: World trade networks

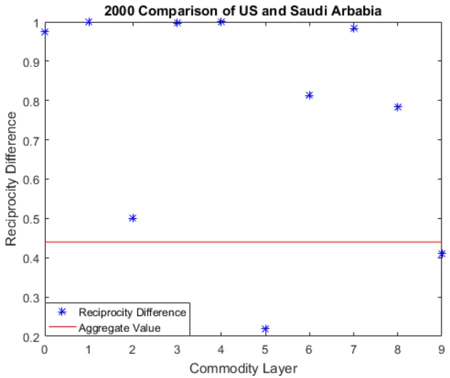
[1] R. Feenstra, R. Lipsey, H. Deng, A.C. Ma, and H. Mo. World Trade Flows: 1962-2000. NBER Working Paper 11040, (2005).

# WTW Layers

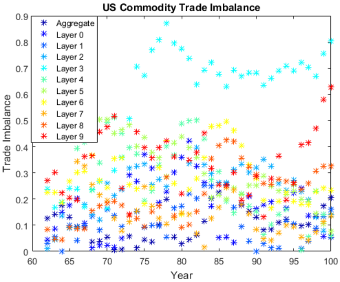
| Layer | Description | Volume | % Total | Transitivity |
|---|---|---|---|---|
| 0 | Food and live animals | 291554437 | 5.1 | .82 |
| 1 | Beverages and tobacco | 48046852 | 0.9 | .67 |
| 2 | Crude materials | 188946835 | 3.3 | .79 |
| 3 | Mineral fuels | 565811660 | 10.0 | .62 |
| 4 | Animal and vegetable oils | 14578671 | 0.3 | .64 |
| 5 | Chemicals | 535703156 | 9.5 | .83 |
| 6 | Manufactured Goods | 790582194 | 13.9 | .87 |
| 7 | Machinery | 2387828874 | 42.1 | .85 |
| 8 | Miscellaneous manufacturing | 736642890 | 13.0 | .83 |
| 9 | Other commodities | 107685024 | 1.9 | .56 |
| All | Aggregate Trade | 5667380593 | 100 | .93 |

Table: Layer information for the 2000 World Trade Web.

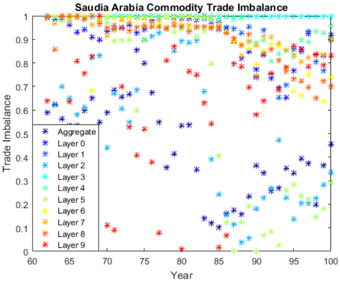
# 2000 USA – SA Trade Imbalances



# USA and SA Commodity Imbalances

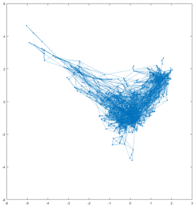



(a) US Commodities

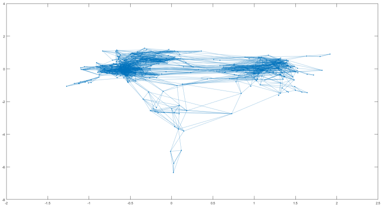



(b) SA Commodities

# Karnataka Village Data[1]



(a) Village 4



(b) Village 61

Figure: Karnataka Villages

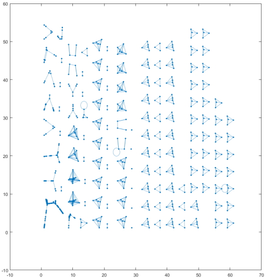
[1] A. Banerjee, A.G. Chandrasekhar, E. Duflo, and M.O. Jackson, The Diffusion of Microfinance. Science, (2013).

# Village Layers

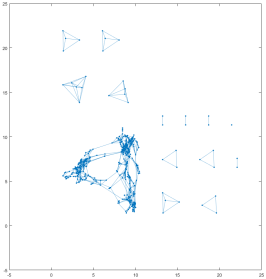
| Layer | Village 4 | | | Village 61 | | |
|---|---|---|---|---|---|---|
| Description | Density | Comp. | Giant % | Density | Comp. | Giant % |
| Borrow Money | .0082 | 26 | .8354 | .0108 | 15 | .9188 |
| Give Advice | .0077 | 49 | .5892 | .0098 | 34 | .7377 |
| Help Make Decisions | .0076 | 61 | .1277 | .0100 | 24 | .8562 |
| Borrow Kerosene or Rice | .0085 | 21 | .8338 | .0113 | 14 | .9171 |
| Lend Kerosene or Rice | .0086 | 22 | .8308 | .0113 | 14 | .9255 |
| Lend Money | .0081 | 14 | .7908 | .0107 | 17 | .9036 |
| Medical Advice | .0075 | 84 | .2938 | .0106 | 14 | .9306 |
| Friends | .0089 | 15 | .9277 | .0105 | 22 | .8714 |
| Relatives | .0085 | 29 | .7231 | .0105 | 26 | .5448 |
| Attend Temple With | .0073 | 117 | .0462 | .0089 | 108 | .0372 |
| Visit Their Home | .0087 | 15 | .9185 | .0116 | 11 | .9475 |
| Visit Your Home | .0088 | 16 | .9108 | .0117 | 11 | .9492 |
| Aggregate | .0121 | 3 | .9862 | .0155 | 8 | .9679 |

Table: Layer information for two of the Karnataka Villages.

# Medical Advice

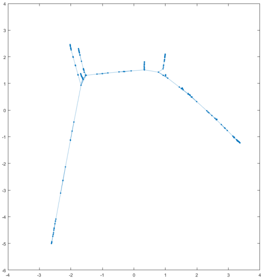



(a) Village 4

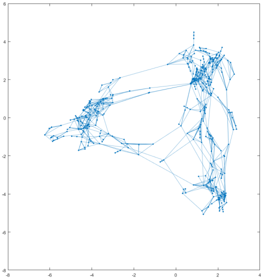



(b) Village 61

# Medical Advice



(a) Village 4



(b) Village 61

# Disjoint Layers

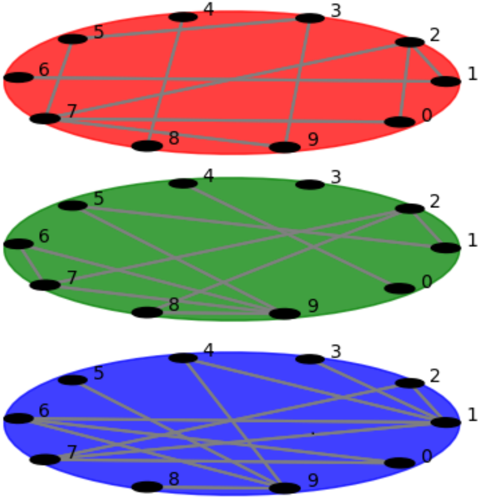
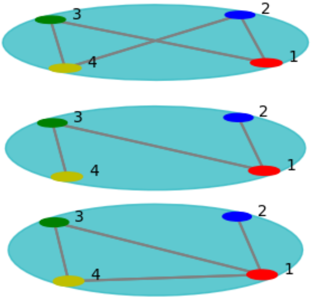
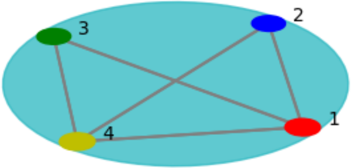



Figure: Disjoint Layers

# Aggregate Models

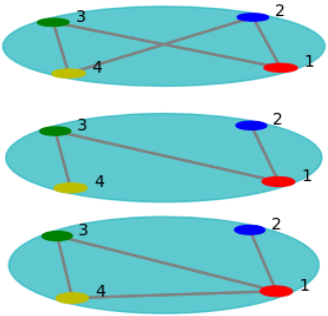



(a) Disjoint Layers

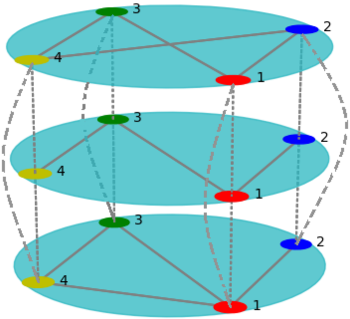



(b) Aggregate

# Matched Sum




(a) Disjoint Layers




(b) Matched Sum

# Algebraic Structure

We can represent the matched sum with a supra–adjacency matrix:

$$\begin{bmatrix} A^1 & wI_n & \cdots & wI_n & wI_n \\ wI_n & A^2 & \cdots & wI_n & wI_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ wI_n & wI_n & \cdots & A^{k-1} & wI_n \\ wI_n & wI_n & \cdots & wI_n & A^k \end{bmatrix}$$

where the $A^\alpha$ are the adjacency matrices of the individual layers and $w$ is a connection strength parameter.

# Strucutral Asymptotics

As the number of layers grows, what happens to the:

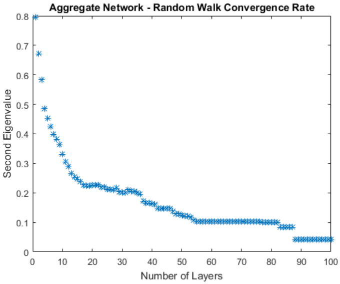
- Density?
- Degree Distribution?
- Transitivity?
- Average Path Length?
- Diameter?
- Clique Number?
- ...

# Strucutral Asymptotics

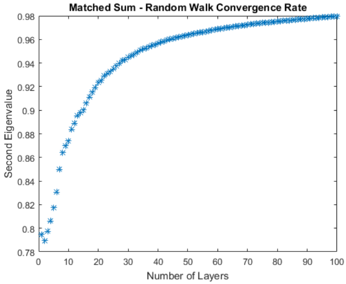
As the number of layers grows, what happens to the:

- Density?
- Degree Distribution?
- Transitivity?
- Average Path Length?
- Diameter?
- Clique Number?
- ...
- **Dynamics?!?**

# Random Walk Convergence




(a) Aggregate




(b) Matched Sum

# Dynamics on Multiplex Networks

- Two types of interactions
    - Within the individual layers
    - Between the layers
- Effects should "pass through" nodes
- Two step iterative model

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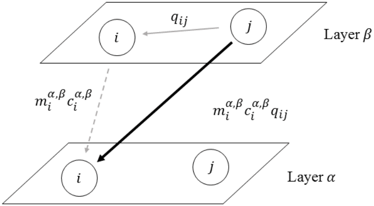


# Matrix Realization

The matrix associated to the total operator also takes a convenient block form:

$$\begin{bmatrix} C_1D_1 & C_1D_2 & \cdots & C_1D_k \\ C_2D_1 & C_2D_2 & \cdots & C_2D_k \\ \vdots & \vdots & \vdots & \vdots \\ C_kD_1 & C_kD_2 & \cdots & C_kD_k \end{bmatrix}$$

Where the $\{D_i\}$ are the dynamical operators associated to the layers and the $\{C_i\}$ are the diagonal proportionality matrices.

# Choice of Coefficients

- Equidistribution
  - $c_i^{\alpha,\beta} = \frac{1}{k}$
  - $C^{\alpha,\beta} = \frac{1}{k} I$
  - Starting Point
- Ranked Layers
  - $c_i^{\alpha,\beta} = c^{\alpha}$
  - $C^{\alpha,\beta} = c^{\alpha} I$
  - Villages
- Unified Node
  - $c_i^{\alpha,\beta} = c_i^{\alpha}$
  - $C^{\alpha,\beta} = C^{\alpha}$
  - WTW
- General Model
  - $c_i^{\alpha,\beta} = c_i^{\alpha,\beta}$
  - Anything goes

# Multiplex Random Walks

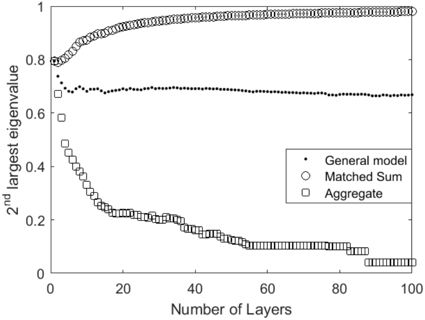



Figure: Comparison of random walk convergence for multiplex models.

# Laplacian Dynamics

Under our dynamical model, where effects pass through node copies to other layers, the heat diffusion interpretation of the Laplacian can be derived from first principles:

$$\frac{dv_i^\alpha}{dt} = -K \sum_{\beta=1}^{k} c_i^{\alpha,\beta} \sum_{n_i^\beta \sim n_j^\beta} (v_i^\beta - v_j^\beta).$$

$$\frac{dv_i^\alpha}{dt} = -K \sum_{\beta=1}^{k} c_i^{\alpha,\beta} (Lv)_i^\beta,$$

# Laplacian Eigenvalue Bounds

Let $\{\lambda_i\}$ be the eigenvalues of $\mathscr{D}$ and $\{\lambda_i^\alpha\}$ be the eigenvalues of the $\alpha$–layer Laplacian $D^\alpha$. We have the following bounds for ranked layers model:

- Fiedler Value:

$$\max_\alpha(\lambda_F^\alpha) \leq k\lambda_F \leq \lambda_F^m + \sum_{\beta \neq m} \lambda_1^\beta$$

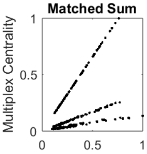
- Leading Value:

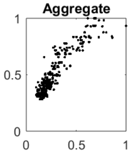
$$\max_i(\lambda_1^i) \leq k\lambda_1 \leq \sum_i \lambda_1^i$$

- General Form:

$$\max_i(\lambda_{n-j}^i) \leq k\lambda_{n-j} \leq \min_{J \vdash n+k-(j+1)} \left( \min_{\sigma \in S_n} \left( \sum_{\alpha=1}^{k} \lambda_{j_\alpha}^{\sigma(\alpha)} \right) \right)$$

# Centrality Comparison



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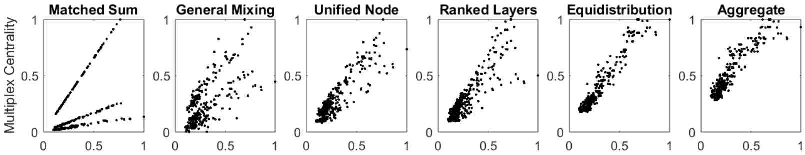

# Multiplex References


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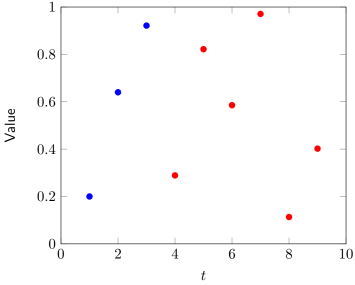
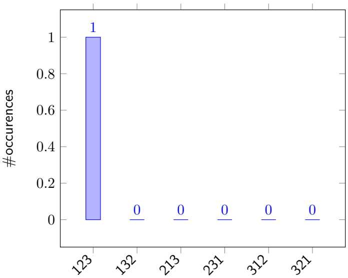
- D. DeFord: *Multiplex Dynamics on the World Trade Web*, Proc. 6th International Conference on Complex Networks and Applications, Studies in Computational Intelligence, Springer, 1111–1123, (2018).

# Complex Time Series



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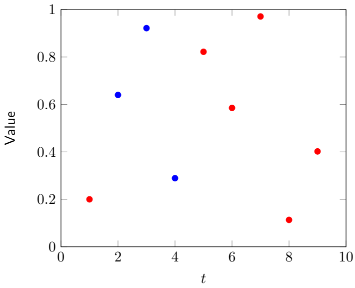


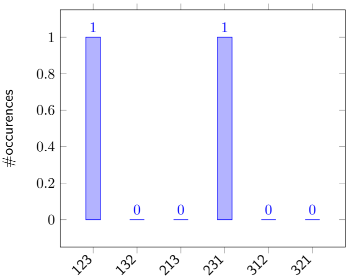

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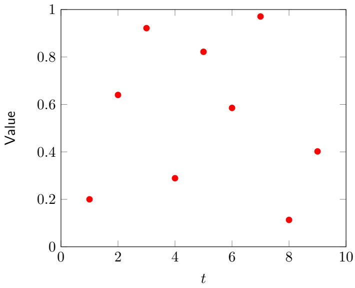
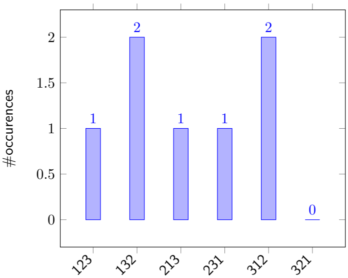
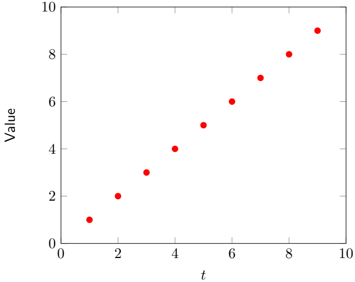
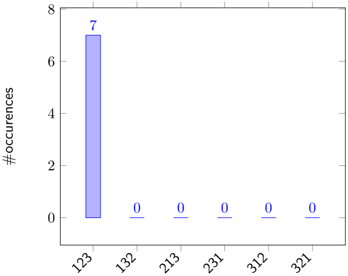


# Simple Time Series



# Complexity Measures

**Definition (Normalized Permutation Entropy)**

$$NPE(\{X_i\}) = -\frac{1}{\log(N!)} \sum_{\pi \in S_n} p_\pi \log(p_\pi)$$

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$$D_{KL}(\{X_i\}||\text{uniform}) = \sum_{\pi \in S_n} p_\pi \log\left(\frac{p_\pi}{\frac{1}{n!}}\right)$$

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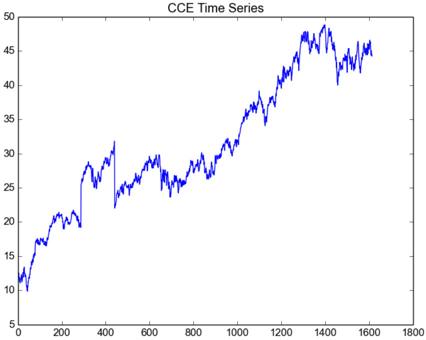
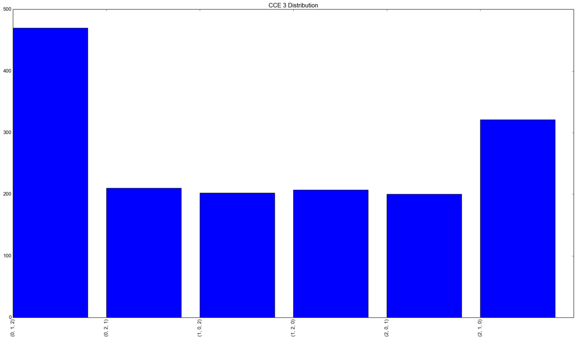
**Observation**

$$1 - NPE(\{X_i\}) = \frac{1}{\log(N!)} D_{KL}(\{X_i\}||uniform)$$

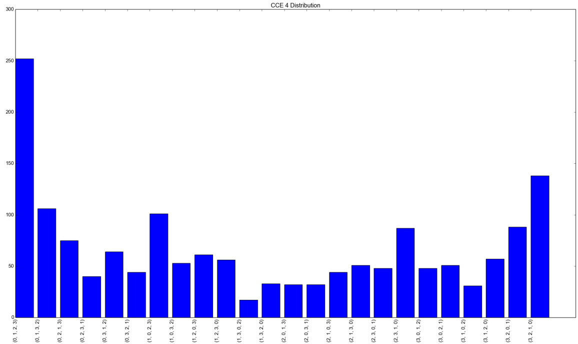
# Stock Data (Closing Prices)



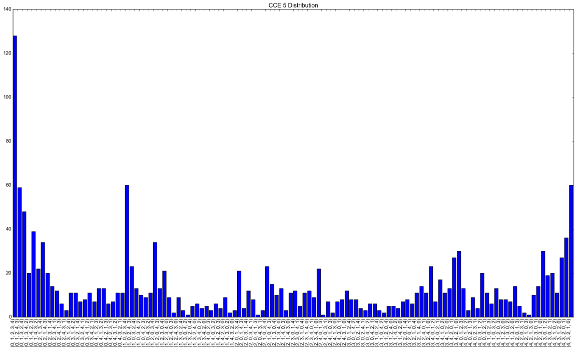
# Stock Data (n=3)



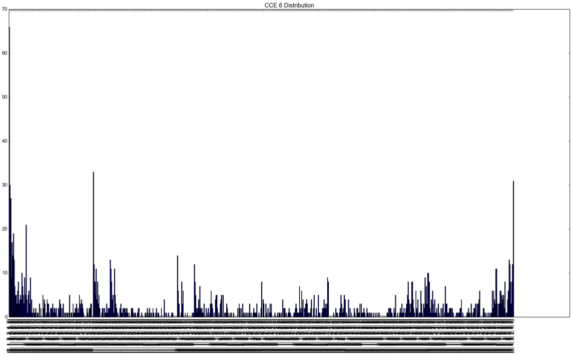
# Stock Data (n=4)



# Stock Data (n=5)



# Stock Data (n=6)



# Random Walk Null Models

## Definition (Random Walk)

Let $\{X_i\}$ be a set of I.I.D. random variables and define $\{Z_i\}$ by $Z_j = \sum_{i=0}^{j} X_j$.

## Lemma

*If $\{Z_i\}$ are defined as above then either $123\ldots n$ or $n(n-1)(n-2)\ldots 1$ occurs with the highest probability.*

## Corollary

*If $\{Z_i\}$ are defined as above and $n \geq 3$ then the expected distribution of permutations is not uniform.*

# New Complexity Measure

**Definition (Null Model KL Divergence)**

$$\mathsf{D_{KL}}_n(X) := \mathsf{D_{KL}}_n(X||Z) = \sum_{\pi \in \mathcal{S}_n} p_\pi \log\left(\frac{p_\pi}{q_\pi}\right),$$

where $p_\pi$ is the relative frequency of $\pi$ in $X$ and $q_\pi$ is the relative frequency of $\pi$ in $Z$.

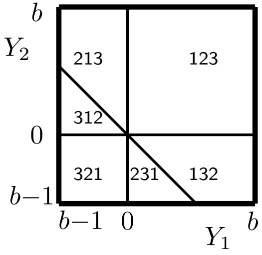
# Hyperplanes

## Example

In order for the pattern $1342$ to appear in the random walk time series we need the following inequalities to hold:

- $X_1 > 0$
- $X_2 > 0$
- $X_3 < 0$
- $X_3 > X_2$
- $X_3 < X_1 + X_2$

# Integration Regions



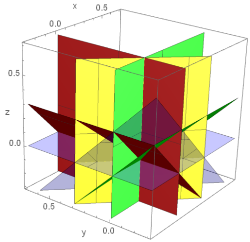




Figure: The regions of integration for patterns in uniform random walks for (**a**) $n = 3$ and (**b**) $n = 4$, sketched here for $b = 0.65$.

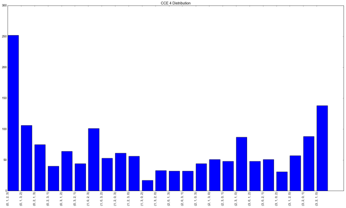
# Null Distributions ($n = 3$)

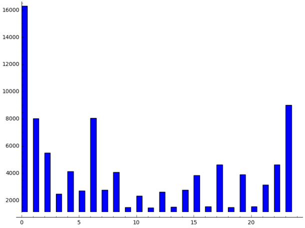
| **Pattern** | **Normal: $\mu = 0$** | **Uniform: $\mu = 0$** | **Uniform: $\mathbb{P}(Y > 0) = b$** |
|---|---|---|---|
| {123} | $1/4$ | $1/4$ | $b^2$ |
| {132, 213} | $1/8$ | $1/8$ | $(1/2)(1-b)^2$ |
| {231, 312} | $1/8$ | $1/8$ | $(1/2)(b^2+2b-1)$ |
| {321} | $1/4$ | $1/4$ | $(1-b)^2$ |

# Null Distributions ($n = 4$)

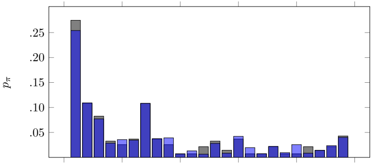
| Pattern | Normal: $\mu = 0$ | Uniform: $\mu = 0$ | Uniform: $\mathbb{P}(Y > 0) = b$ |
|---|---|---|---|
| {1234} | 0.1250 | 1/8 | $b^3$ |
| {1243, 2134} | 0.0625 | 1/16 | $(1/2)b(1-b)(3b-1)$ |
| {1324} | 0.0417 | 1/24 | $(1/3)(1-b)(7b^2-5b+1)$ |
| {1342, 3124} | 0.0208 | 1/24 | $(1/6)(1-b)^2(4b-1)$ |
| {1423, 2314} | 0.0355 | 1/48 | $(1/6)(1-b)^2(5b-2)$ |
| {1432, 2143, 3214} | 0.0270 | 1/48 | $\begin{cases}(1/6)(2-24b+48b^2-15b^3) & \text{if } b \leq 2/3 \\ (b-1)^2(2b-1) & \text{if } b > 2/3\end{cases}$ |
| {2341, 3412, 4123} | 0.0270 | 1/48 | $(1/6)(1-b)^3$ |
| {2413} | 0.0146 | 1/48 | $(1/6)(1-b)^3$ |
| {2431, 4213} | 0.0208 | 1/24 | $\begin{cases}(1/6)(24b^3-45b^2+27b-5) & \text{if } b \leq 2/3 \\ \hline (1/2)(1-b)^3 & \text{if } b > 2/3\end{cases}$ |
| {3142} | 0.0146 | 1/48 | $\begin{cases}(1/6)(25b^3-48b^2+30b-6) & \text{if } b \leq 2/3 \\ (1/3)(1-b)^3 & \text{if } b > 2/3\end{cases}$ |
| {3241, 4132} | 0.0355 | 1/48 | $(1/6)(1-b)^3$ |
| {3421, 4312} | 0.0625 | 1/16 | $(1/2)(1-b)^3$ |
| {4231} | 0.0417 | 1/24 | $(1/3)(1-b)^3$ |
| {4321} | 0.1250 | 1/8 | $(1-b)^3$ |

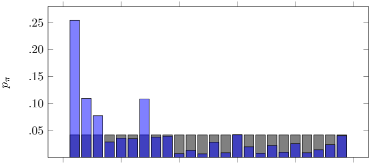
# Uniform Steps CCE



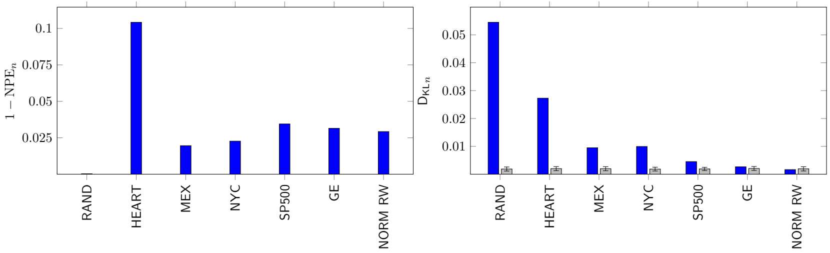



# Uniform Steps S&P 500



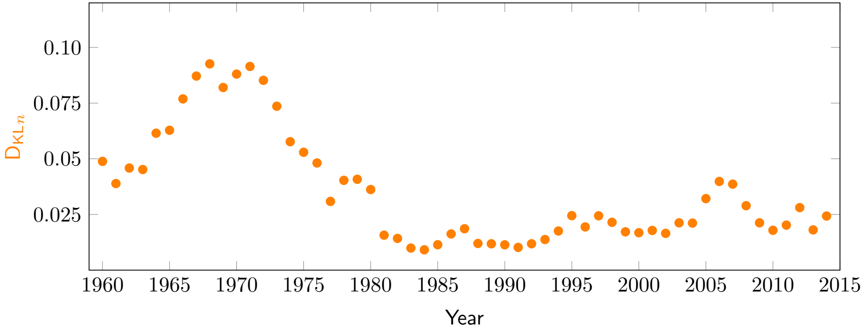



# Data Comparisons




(a) (b)

# Stock Market Example



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# That's all...

Thank You!