Dynamics on Multiplex Networks

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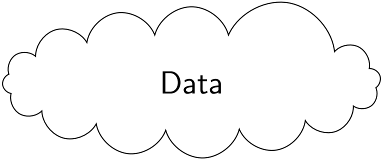
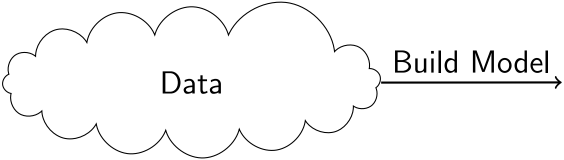
Outline

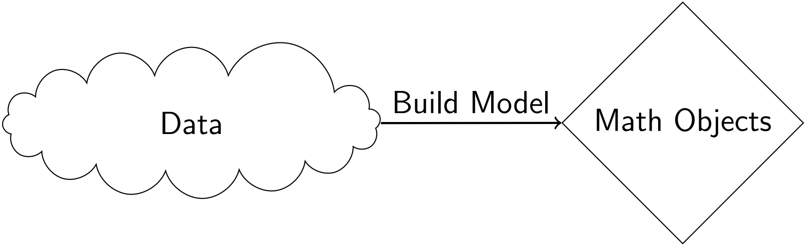
Philosophy



Philosophy



Philosophy



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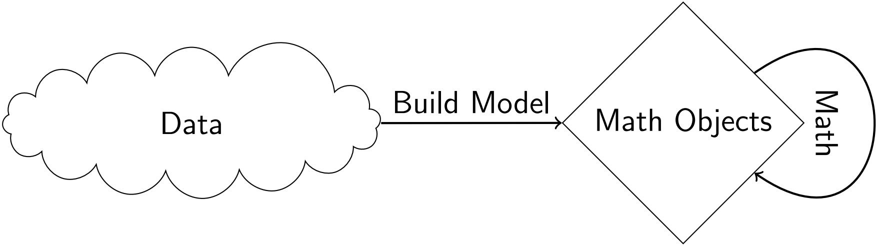
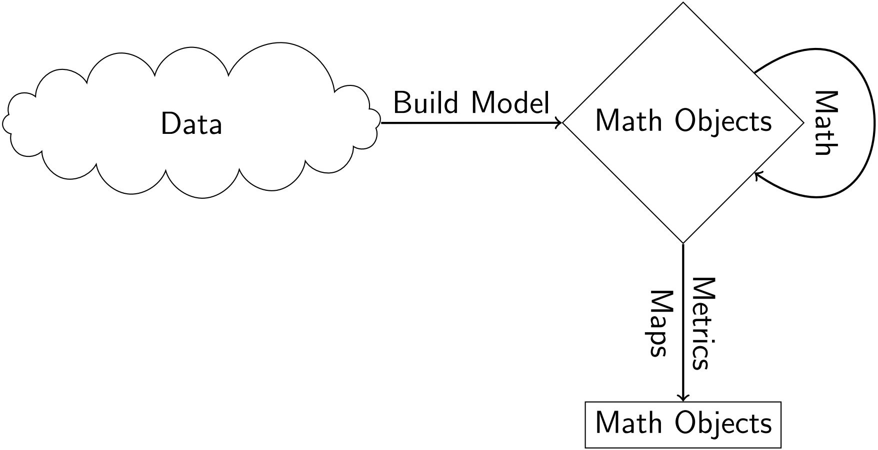
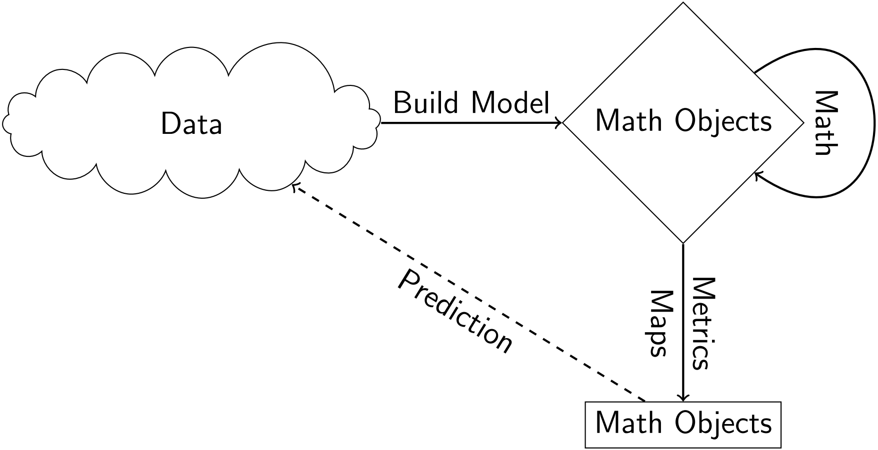


Philosophy


Data
Build Model
Math Objects
Math
Maps
Metrics
Math Objects

Philosophy



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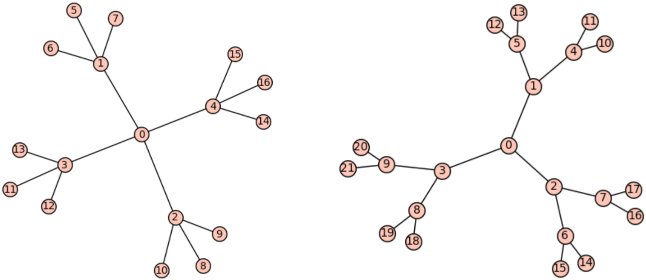
Adjacency Spectra of Regular Trees



(a) $X_2^{(4)}$ (b) $X_3^{(3)}$

Eigenvalues of (3)–trees

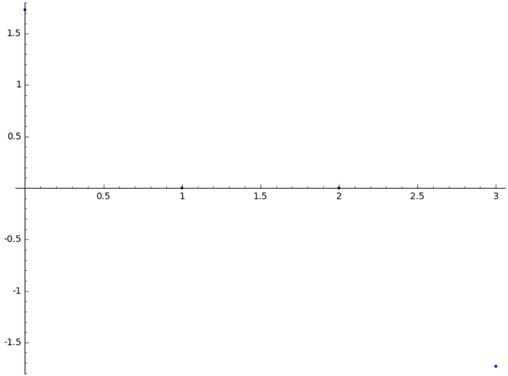


Figure : Eigenvalues of $X_1^{(3)}$

Eigenvalues of (3)–trees

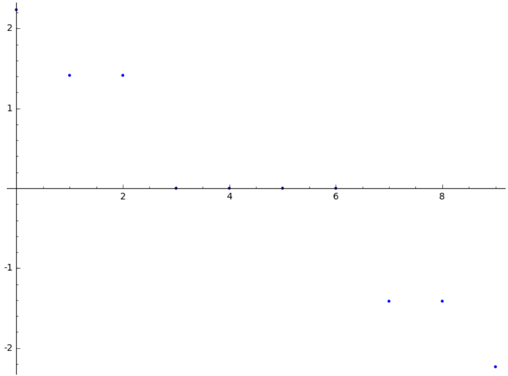


Figure : Eigenvalues of $X_2^{(3)}$

Eigenvalues of (3)–trees

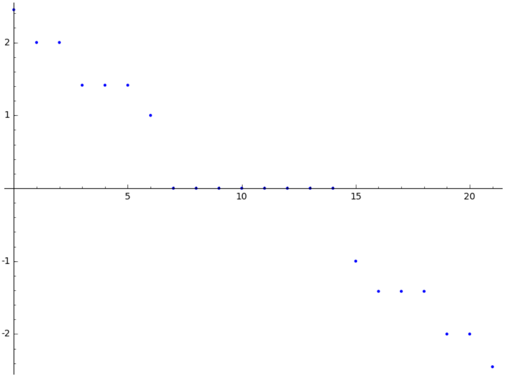


Figure : Eigenvalues of $X_3^{(3)}$

Eigenvalues of (3)–trees

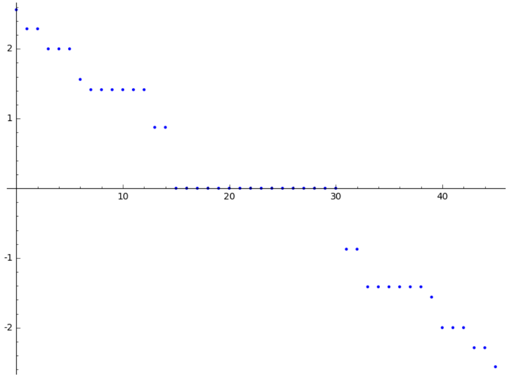


Figure : Eigenvalues of $X_4^{(3)}$

Eigenvalues of (3)–trees

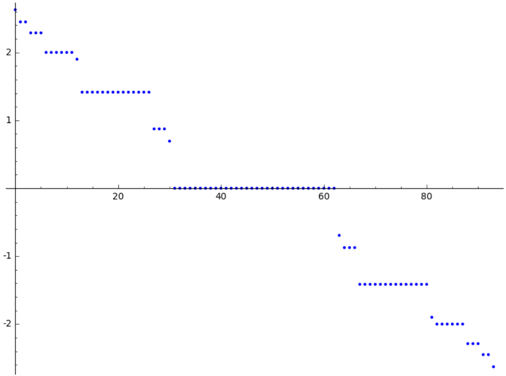


Figure : Eigenvalues of $X_5^{(3)}$

Eigenvalues of (3)–trees

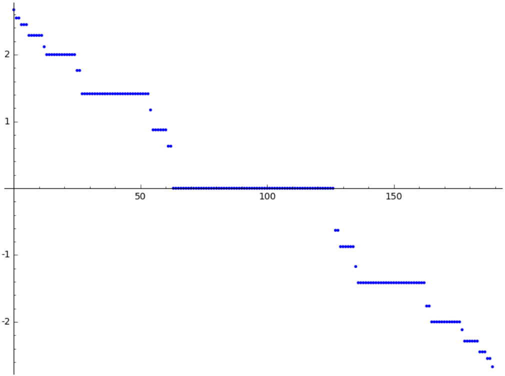


Figure : Eigenvalues of $X_6^{(3)}$

Eigenvalues of (3)–trees

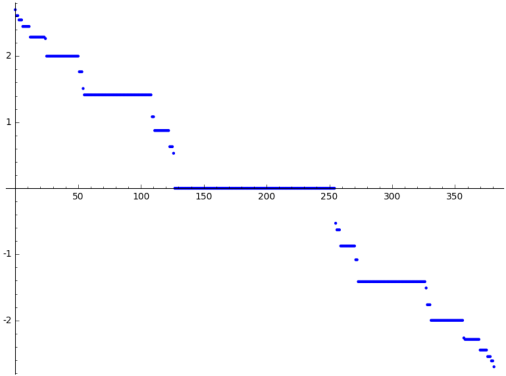


Figure : Eigenvalues of $X_7^{(3)}$

Eigenvalues of (3)–trees

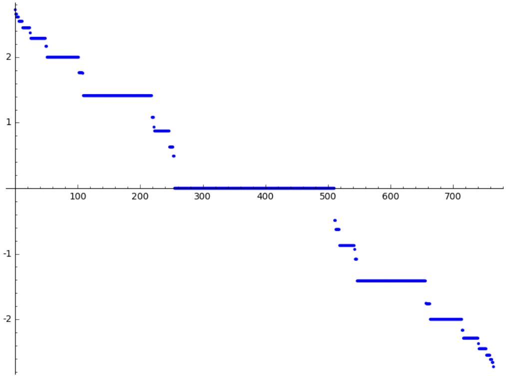


Figure : Eigenvalues of $X_8^{(3)}$

Eigenvalues of Regular Trees

Define two families of polynomials:

$$P_n^k(x) = xP_{n-1}^k - (k-1)P_{n-2}^k$$

with initial conditions $P_0^k(x) = 0$, $P_1^k(x) = 1$, and $P_2^k(x) = x$ and

$$Q_n^k(x) = xP_n^k(x) - kP_{n-1}^k(x).$$

Theorem

The roots of $P_s^k(x)$ for $1 \leq s \leq r$ and $Q_r^k(x)$ are precisely the eigenvalues of the finite k-ary tree X_r^k.

Enumerative Results

Theorem

If λ is a root P_r^k and not a root of P_m^k for any $m < r$ then asymptotically (as $r \longrightarrow \infty$), the proportion of eigenvalues of X_k^r is $\frac{(k-2)^2}{(k-1)^r - 1}$.

Corollary

$$\sum_{n=2}^{\infty} \frac{\varphi(n)(k-2)^2}{(k-1)^n - 1} = 1.$$

Corollary ($k = 3$)

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{2^n - 1} = 2.$$

Tree Questions

Question

Given a graph G and an associated adjacency eigenpair (v, λ) does there exist a subgraph H of G so that $(v|_H, \lambda)$ is an eigenpair for H?

Question

Is there a nice closed form for the endpoints of the Cantor–like sets:

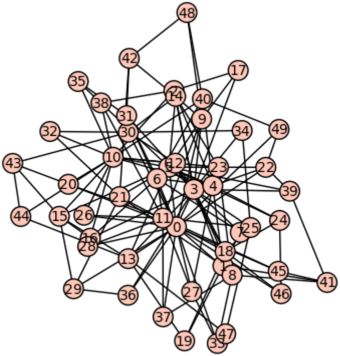
$$\left\{ \sum_{n=1}^{\infty} \frac{(k-2)^2}{(k-1)^n - 1} \sum_{\substack{(\ell,n)=1 \\ \ell < \frac{an}{m}}} 1 : \begin{array}{c} m \in \mathbb{N} \\ (a,m) = 1 \end{array} \right\}$$

Question

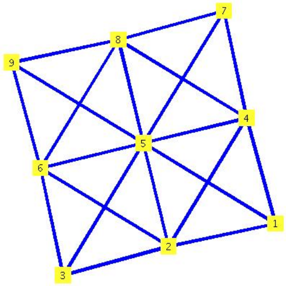
Can we characterize the sequences of graphs $G_1, G_2, \ldots$ satisfying for all $\varepsilon > 0$ there exists a finite set $\Lambda \subset \mathbb{R}$ and a $N \in \mathbb{N}$ such that for all $n > N$:

$$\frac{|\{\lambda \in \operatorname{spec}(G_n) : \lambda \notin \Lambda\}|}{|\operatorname{spec}(G_n)|} < \varepsilon.$$

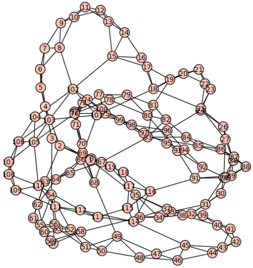
Complex Networks



Examples

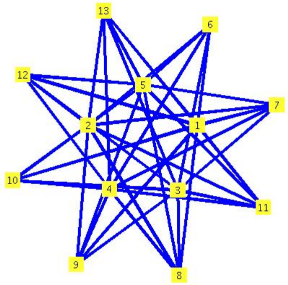


(a) Graph

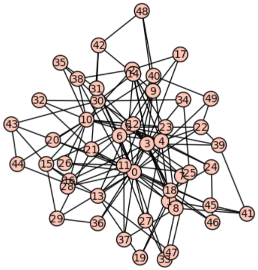


(b) Network

Examples

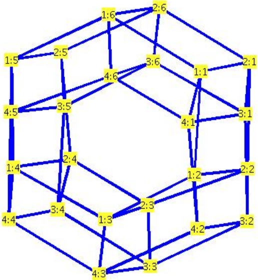


(a) Graph



(b) Network

Examples



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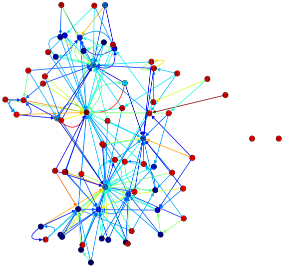
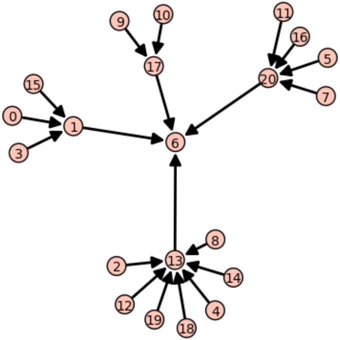
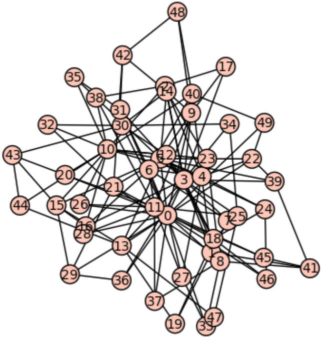


(b) Network

Centrality



Centrality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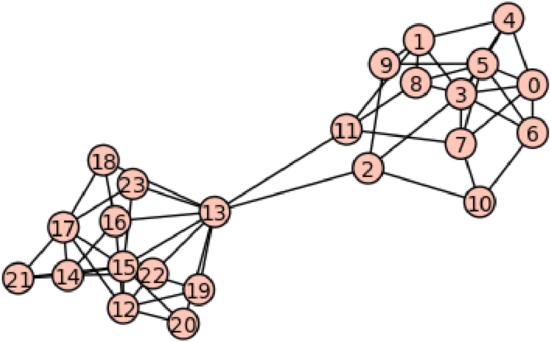
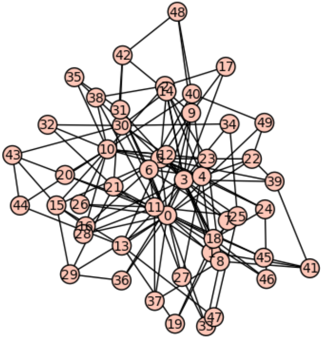
Clustering



Clustering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Operators on Networks

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- Adjacency
  - Symmetric, binary
  - Eigenvector centrality

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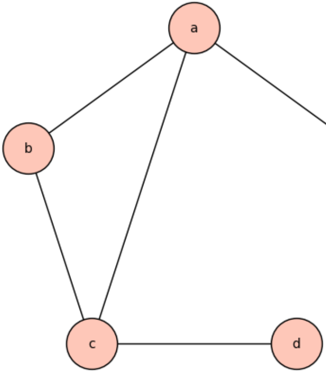
- Laplacian
  - Matrix: $L = D - A$
  - Positive semi–definite
  - Discretized version of Laplacian heat diffusion

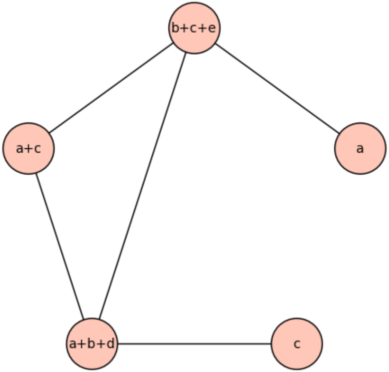
Operators on Networks

- Adjacency
  - Symmetric, binary
  - Eigenvector centrality
- Laplacian
  - Matrix: $L = D - A$
  - Positive semi–definite
  - Discretized version of Laplacian heat diffusion
- Random Walk
  - Matrix; $R = D^{-1}A$
  - Always stochastic, regular if G is connected
  - Transition matrix of associated Markov process

Adjacency Action

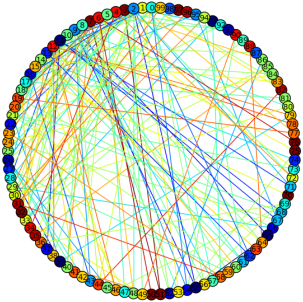


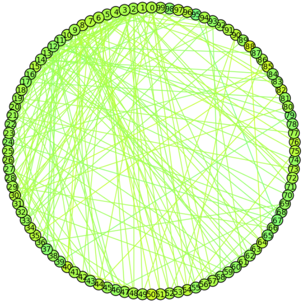
(c) Before



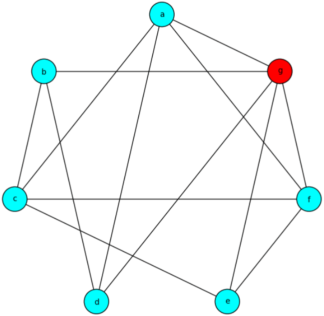
(d) After

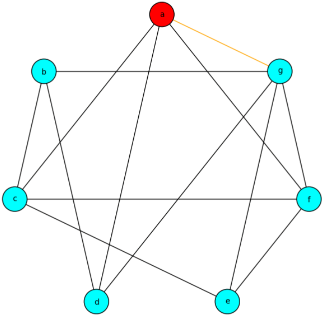
Diffusion Animation





Random Walk Animation





Spectral Graph Theory

Fan Chung: *Spectral Graph Theory*, AMS, (1997).

"Roughly speaking, half of the main problems of spectral theory lie in deriving bounds on the distributions of eigenvalues. The other half concern the impact and consequences of the eigenvalue bounds as well as their applications."

What is a multiplex?

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Definition

A *multiplex* is a collection of graphs all defined on the same node set.

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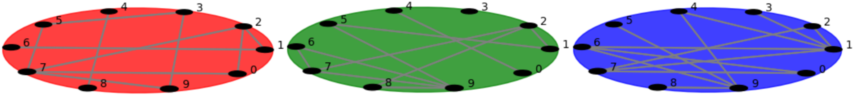
A *multiplex* is a collection of graphs all defined on the same node set.



(a) Family (b) Colleagues (c) Facebook

World Trade Web[1]

Figure : World trade networks

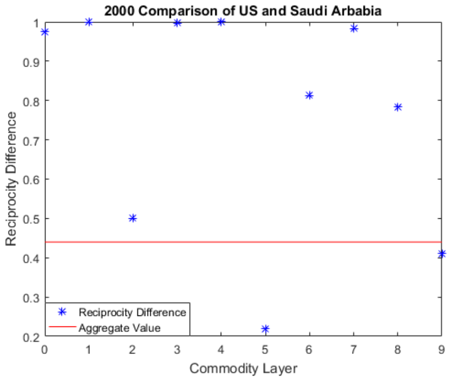
[1] R. Feenstra, R. Lipsey, H. Deng, A.C. Ma, and H. Mo. World Trade Flows: 1962-2000. NBER Working Paper 11040, (2005).

WTW Layers

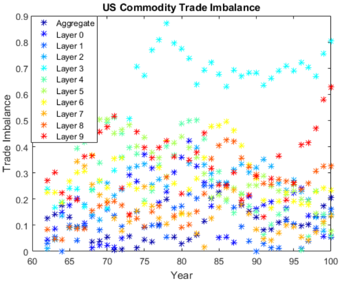
Layer	Description	Volume	% Total	Transitivity
0	Food and live animals	291554437	5.1	.82
1	Beverages and tobacco	48046852	0.9	.67
2	Crude materials	188946835	3.3	.79
3	Mineral fuels	565811660	10.0	.62
4	Animal and vegetable oils	14578671	0.3	.64
5	Chemicals	535703156	9.5	.83
6	Manufactured Goods	790582194	13.9	.87
7	Machinery	2387828874	42.1	.85
8	Miscellaneous manufacturing	736642890	13.0	.83
9	Other commodities	107685024	1.9	.56
All	Aggregate Trade	5667380593	100	.93

Table : Layer information for the 2000 World Trade Web.

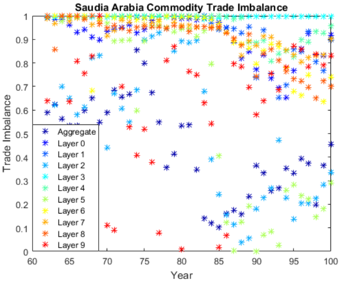
2000 USA – SA Trade Imbalances



USA and SA Commodity Imbalances

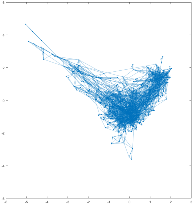


(a) US Commodities

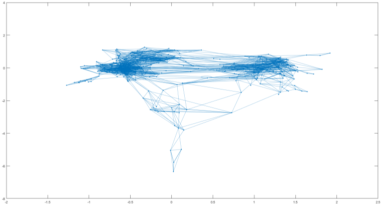


(b) SA Commodities

Karnataka Village Data[1]



(a) Village 4



(b) Village 61

Figure : Karnataka Villages

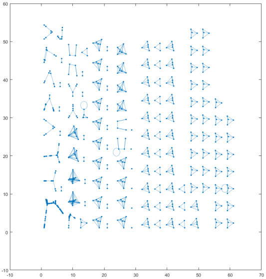
1 A. Banerjee, A.G. Chandrasekhar, E. Duflo, and M.O. Jackson, The Diffusion of Microfinance. Science, (2013).

Village Layers

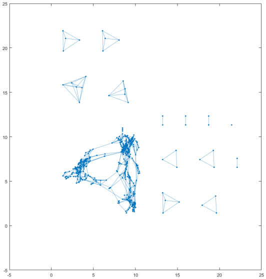
Layer	Village 4			Village 61		
Description	Density	Comp.	Giant %	Density	Comp.	Giant %
Borrow Money	.0082	26	.8354	.0108	15	.9188
Give Advice	.0077	49	.5892	.0098	34	.7377
Help Make Decisions	.0076	61	.1277	.0100	24	.8562
Borrow Kerosene or Rice	.0085	21	.8338	.0113	14	.9171
Lend Kerosene or Rice	.0086	22	.8308	.0113	14	.9255
Lend Money	.0081	14	.7908	.0107	17	.9036
Medical Advice	.0075	84	.2938	.0106	14	.9306
Friends	.0089	15	.9277	.0105	22	.8714
Relatives	.0085	29	.7231	.0105	26	.5448
Attend Temple With	.0073	117	.0462	.0089	108	.0372
Visit Their Home	.0087	15	.9185	.0116	11	.9475
Visit Your Home	.0088	16	.9108	.0117	11	.9492
Aggregate	.0121	3	.9862	.0155	8	.9679

Table : Layer information for two of the Karnataka Villages.

Medical Advice

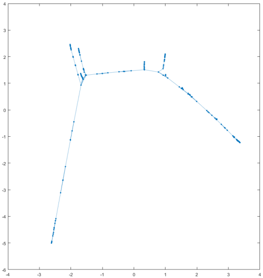


(a) Village 4

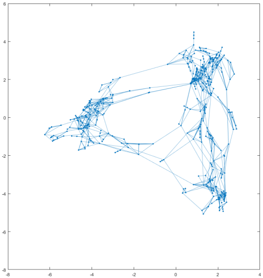


(b) Village 61

Medical Advice



(a) Village 4



(b) Village 61

Disjoint Layers

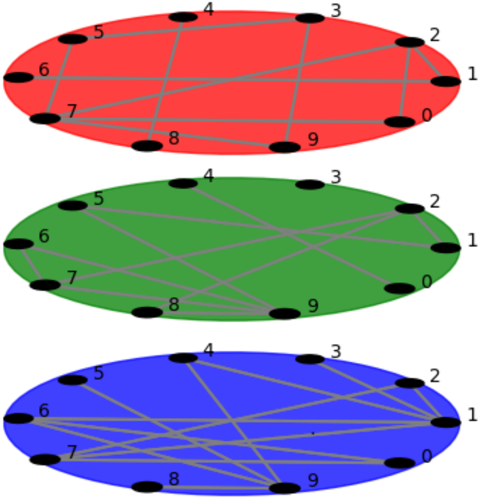
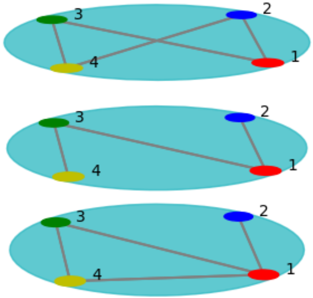
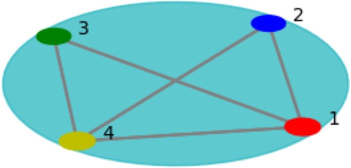


Figure : Disjoint Layers

Aggregate Models

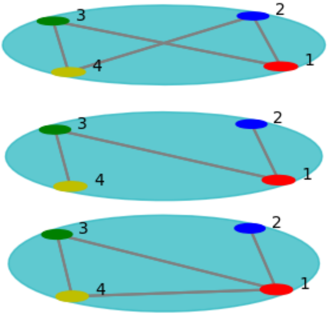


(a) Disjoint Layers

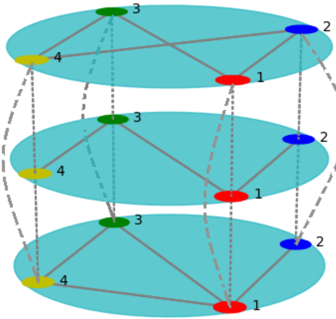


(b) Aggregate

Matched Sum



(a) Disjoint Layers



(b) Matched Sum

Algebraic Structure

We can represent the matched sum with a supra–adjacency matrix:

$$\begin{bmatrix} A^1 & wI_n & \cdots & wI_n & wI_n \\ wI_n & A^2 & \cdots & wI_n & wI_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ wI_n & wI_n & \cdots & A^{k-1} & wI_n \\ wI_n & wI_n & \cdots & wI_n & A^k \end{bmatrix}$$

where the A^α are the adjacency matrices of the individual layers and w is a connection strength parameter.

Strucutral Asymptotics

As the number of layers grows, what happens to the:

- Density?
- Degree Distribution?
- Transitivity?
- Average Path Length?
- Diameter?
- Clique Number?
- ...

Structutral Asymptotics

As the number of layers grows, what happens to the:

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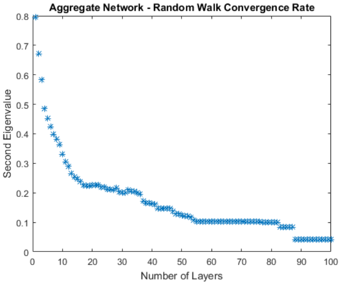
- Diameter?
- Clique Number?
- ...
- **Dynamics?!?**

Random Walk Convergence



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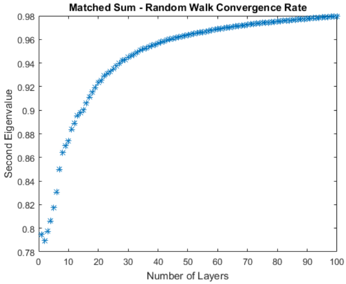

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- Two types of interactions
  - Within the individual layers
  - Between the layers
- Effects should "pass through" nodes
- Two step iterative model

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- Symbolically:

$$v' = \mathscr{D}v$$

$$(v')_i^{\alpha} = \sum_{\beta=1}^{k} m_i^{\alpha,\beta} c_i^{\alpha,\beta} (Dv)_i^{\beta}$$

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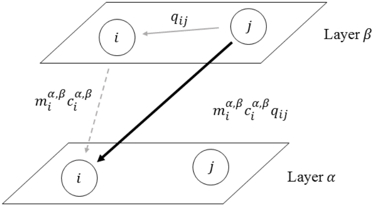
- Two step iterative model
- Symbolically:

$$v' = \mathscr{D}v$$

$$(v')_i^{\alpha} = \sum_{\beta=1}^{k} m_i^{\alpha,\beta} c_i^{\alpha,\beta} (Dv)_i^{\beta}$$



Matrix Realization

The matrix associated to the total operator also takes a convenient block form:

$$\begin{bmatrix} C_1D_1 & C_1D_2 & \cdots & C_1D_k \\ C_2D_1 & C_2D_2 & \cdots & C_2D_k \\ \vdots & \vdots & \vdots & \vdots \\ C_kD_1 & C_kD_2 & \cdots & C_kD_k \end{bmatrix}$$

Where the $\{D_i\}$ are the dynamical operators associated to the layers and the $\{C_i\}$ are the diagonal proportionality matrices.

Choice of Coefficients

- Equidistribution
  - $c_i^{\alpha,\beta} = \frac{1}{k}$
  - $C^{\alpha,\beta} = \frac{1}{k} I$
  - Starting Point/Aggregate

Choice of Coefficients

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- Ranked Layers
  - $c_i^{\alpha,\beta} = c^{\alpha}$
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  - Villages
- Unified Node
  - $c_i^{\alpha,\beta} = c_i^{\alpha}$
  - $C^{\alpha,\beta} = C^{\alpha}$
  - WTW

Choice of Coefficients

- Equidistribution
  - $c_i^{\alpha,\beta} = \frac{1}{k}$
  - $C^{\alpha,\beta} = \frac{1}{k} I$
  - Starting Point/Aggregate
- Ranked Layers
  - $c_i^{\alpha,\beta} = c^\alpha$
  - $C^{\alpha,\beta} = c^\alpha I$
  - Villages
- Unified Node
  - $c_i^{\alpha,\beta} = c_i^\alpha$
  - $C^{\alpha,\beta} = C^\alpha$
  - WTW
- General Model
  - $c_i^{\alpha,\beta} = c_i^{\alpha,\beta}$
  - Anything goes/Matched Sum

Multiplex Random Walks

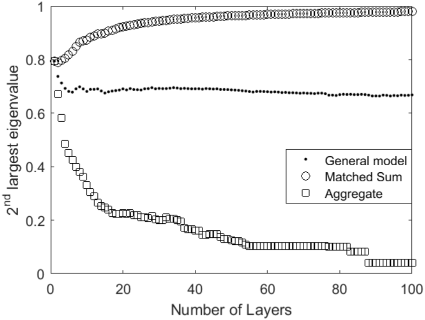
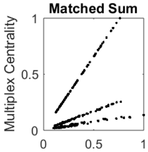
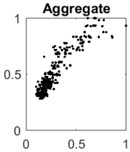


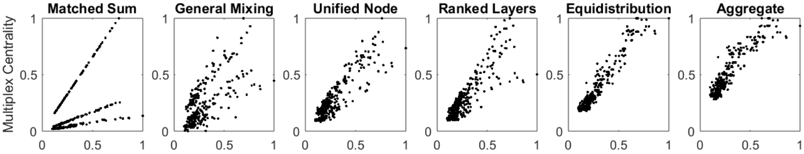
Figure : Comparison of random walk convergence for multiplex models.

Centrality Comparison





Centrality Comparison



Global Aggregate Rankings

Year	1970	1980	1990	2000
1	US	US	US	US
2	Germany	Germany	Germany	Germany
3	Canada	Japan	Japan	Japan
4	UK	UK	France	China
5	Japan	France	UK	UK
6	France	Saudi Arabia	Italy	France

Table : RWBC values for the aggregate WTW.

Full Multiplex RWBC

Ranking	Country	Layer
1	US	7
2	Germany	7
3	China	7
4	UK	7
5	Japan	7
6	US	8
7	Canada	7
8	France	7
9	Japan	3
10	US	6
12	US	3
13	Netherlands	7
14	Germany	6
15	Italy	7

Table : Multiplex RWBC values for the 2000 WTW.

Commodity Appearance

Layer	Ranking	Country
0	22	Japan
1	199	Germany
2	47	China
3	9	Japan
4	184	Australia
5	23	Germany
6	10	US
7	1	US
8	6	US
9	39	US

Table : First appearance of each layer in the rankings.

Ranking Movement

Layer 7 Ranking	Country	Multiplex Ranking
1	USA	1
2	Japan	5
3	Germany	2
4	China	3
5	France	8
6	UK	4
7	South Korea	18
8	Canada	7
9	Malaysia	16
10	Mexico	20

Table : Comparison of the relative rankings of the RWBC on Layer 7 versus the multiplex RWBC.

Laplacian Dynamics

Under our dynamical model, where effects pass through node copies to other layers, the heat diffusion interpretation of the Laplacian can be derived from first principles:

$$\frac{dv_i^\alpha}{dt} = -K \sum_{\beta=1}^{k} c_i^{\alpha,\beta} \sum_{n_i^\beta \sim n_j^\beta} (v_i^\beta - v_j^\beta).$$

$$\frac{dv_i^\alpha}{dt} = -K \sum_{\beta=1}^{k} c_i^{\alpha,\beta} (Lv)_i^\beta,$$

Laplacian Eigenvalue Bounds

Let $\{\lambda_i\}$ be the eigenvalues of $\mathscr{D}$ and $\{\lambda_i^\alpha\}$ be the eigenvalues of the α–layer Laplacian D^α. We have the following bounds for ranked layers model:

- Fiedler Value:

$$\max_\alpha(\lambda_F^\alpha) \leq k\lambda_F \leq \lambda_F^m + \sum_{\beta \neq m} \lambda_1^\beta$$

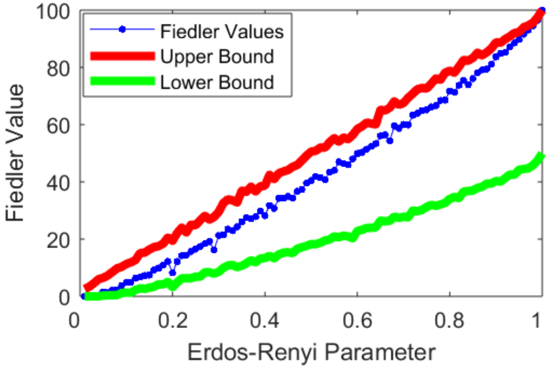
- Leading Value:

$$\max_i(\lambda_1^i) \leq k\lambda_1 \leq \sum_i \lambda_1^i$$

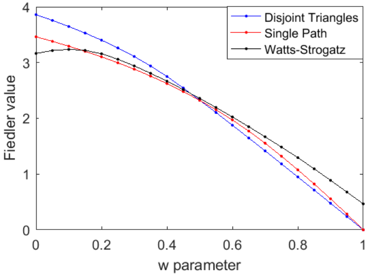
- General Form:

$$\max_i(\lambda_{n-j}^i) \leq k\lambda_{n-j} \leq \min_{J \vdash n+k-(j+1)} \left(\min_{\sigma \in S_n} \left(\sum_{\alpha=1}^{k} \lambda_{j_\alpha}^{\sigma(\alpha)} \right) \right)$$

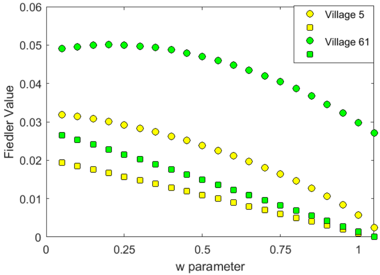
Bounds Example



Social Diffusion



(a) Synthetic



(b) Villages

Network Clustering Formulation

$$s(i) = \begin{cases} 1 & \text{if} i \in \mathscr{B} \\ -1 & \text{if} i \in \mathscr{C} \end{cases}.$$

$$\begin{aligned} cut(\mathscr{B}, \mathscr{C}) &= \sum_{p \in \mathscr{B}, q \in \mathscr{C}} A_{pq} \\ &= \sum_{p,q=1}^{n} A_{pq} \frac{1}{2}(1 - s(p)s(q)) \\ &= \frac{1}{2}\left(\sum_{i=1}^{n} deg(i) - \frac{1}{2}\sum_{i,j=1}^{n} s(i)(A_{ij} + A_{ji})s(j)\right) \\ &= \frac{1}{2}\left(\sum_{i,j=1}^{n} s(i)deg(i)s(j)\delta_{ij} - \frac{1}{2}\sum_{i,j=1}^{n} s(i)(A_{ij} + A_{ji})s(j)\right) \\ &= \frac{1}{2}(s^T D s - \frac{1}{2}s^T(A + A^T)s) = \frac{1}{2}s^T L s \end{aligned}$$

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Multiplex Cut Formulations

In order to generalize this approach to the multiplex setting we have to address several questions:

- Must copies of the same node cluster together?
- How much should we weight the interlayer edges?
- How should the weight asymmetry affect the outcome?

For both models we have analogues of the adjacency and Laplacian matrices but the resulting clustering optimization is much more complicated.

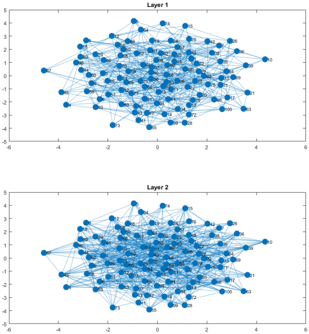
Matched Sum

$$
\begin{aligned}
&\bar{s}^T \mathcal{L}_{\mathfrak{A}} \bar{s} \\
&= \bar{s}^T \begin{pmatrix} \mathcal{L}_{\frac{1}{2}(A^1+(A^1)^T)} + (k-1)I\mathfrak{w} & \dots & -\mathfrak{w}I \\ \vdots & \ddots & \vdots \\ -\mathfrak{w}I & \dots & \mathcal{L}_{\frac{1}{2}(A^k+(A^k)^T)} - (k-1)I \end{pmatrix} \bar{s} \\
&= \sum_{\alpha=1}^{k} \left(s_\alpha^T \mathcal{L}_{\frac{1}{2}(A^\alpha+(A^\alpha)^T)} s_\alpha + (k-1)\mathfrak{w} s_\alpha^T s_\alpha \right) - \sum_{\alpha=1}^{k} \sum_{\beta \neq \alpha} \mathfrak{w} s_\alpha^T s_\beta \\
&= \sum_{\alpha=1}^{k} s_\alpha^T \mathcal{L}_{\frac{1}{2}(A^\alpha+(A^\alpha)^T)} s_\alpha + k^2 n \mathfrak{w} - \mathfrak{w} \sum_{\alpha,\beta=1}^{k} s_\alpha^T s_\beta .
\end{aligned}
$$

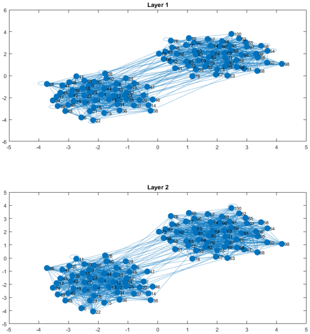
Dynamical Model

$$
\begin{aligned}
\bar{s}^T \mathcal{L}_{\mathfrak{A}} \bar{s} &= \sum_{\alpha,\beta=1}^{k} s_\alpha^T (\mathfrak{D}^{\alpha,\beta} - B^{\alpha,\beta}) s_\beta \\
&= \sum_{\alpha}^{k} s_\alpha^T \mathfrak{D}^{\alpha,\alpha} s_\alpha - \sum_{\alpha,\beta=1}^{k} s_\alpha^T B^{\alpha,\beta} s_\beta \\
&= \sum_{\alpha} s_\alpha^T \mathcal{L}_{B^{\alpha,\alpha}} s_\alpha + \frac{1}{2} \sum_{\alpha \neq \beta} s_\alpha^T (deg_{in}^{\alpha,\beta} + deg_{out}^{\alpha,\beta}) s_\alpha - \sum_{\alpha \neq \beta} s_\alpha^T B^{\alpha,\beta} s_\beta \\
&= \sum_{\alpha} s_\alpha^T \mathcal{L}_{B^{\alpha,\alpha}} s_\alpha + \sum_{\alpha \neq \beta} \left(M^{\alpha,\beta} - s_\alpha^T B^{\alpha,\beta} s_\beta \right),
\end{aligned}
$$

Experiment Models

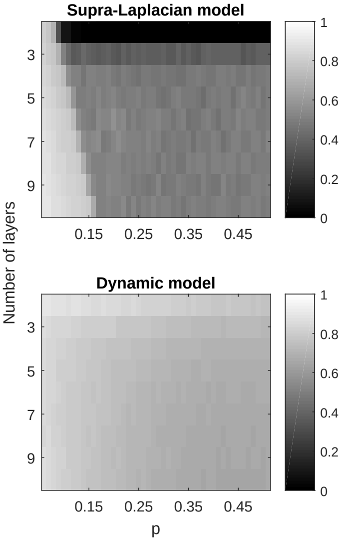


(a) Erdos–Renyi Layers

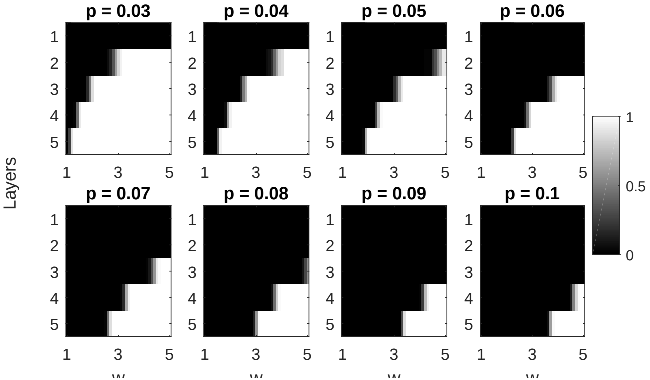


(b) SBM Layers

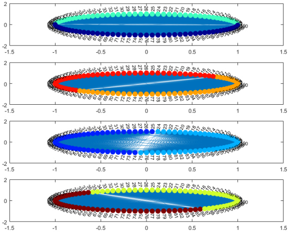
Erdos–Renyi Layers



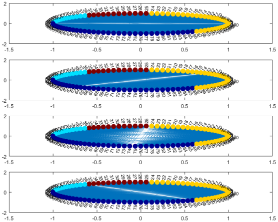
SBM Layers



Clustering Comparison



(a) Matched Sum



(c) Aggregate

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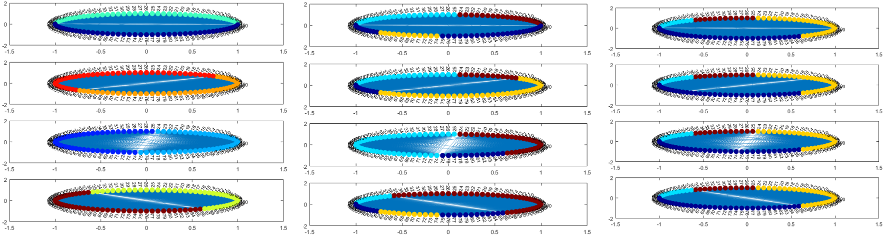
Clustering Comparison



(a) Matched Sum (b) Dynamical Model (c) Aggregate

Multiplex Questions

Question (Data Examples)

- *Social Networks*
- *Transportation Networks*
- *Paired Networks*

Question (How to account for layer heterogeneity?)

- *Structural Consequences*
- *Dynamical Consequences*
- *Spectral Consequences*

Question (How to account for node indivisibility?)

- *Merge Centrality Measures*
- *Define Neighborhoods*
- *Multi–Membership Communities*

Multiplex References

- M. Kivela, A. Arenas, M. Barthelemy, J. Gleeson, Y. Moreno, and M. Porter: *Multilayer Networks*, Journal of Complex Networks, 1–69, (2014).
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That's all...

Thank You!