Dynamics on Multiplex Networks

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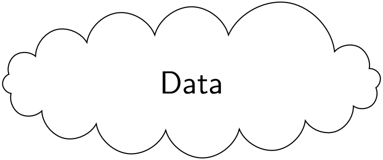
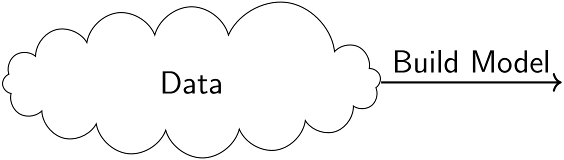
Outline

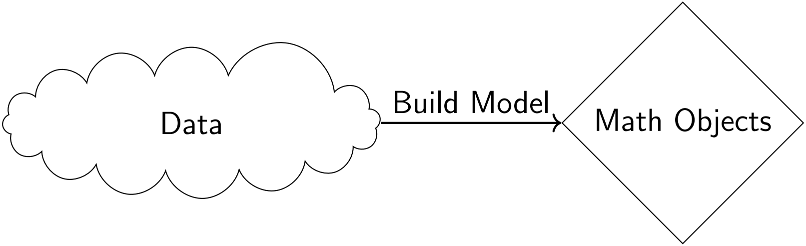
Philosophy



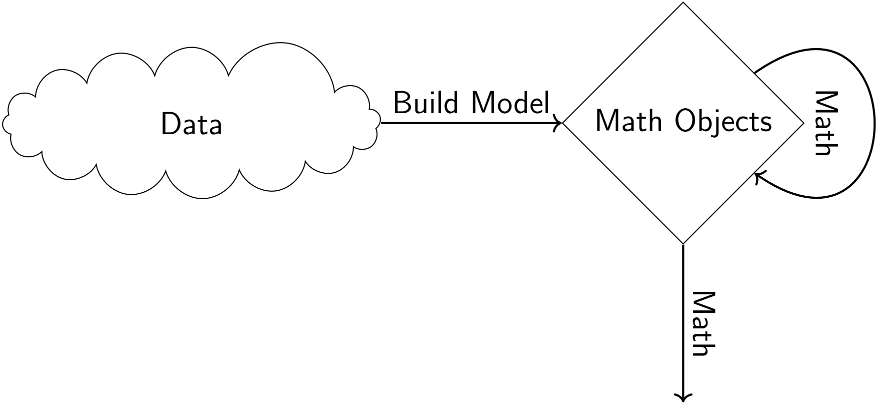
Philosophy



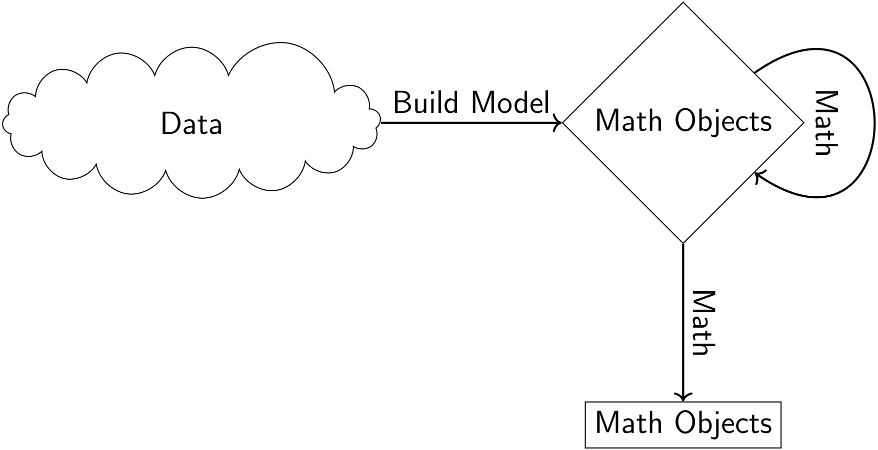
Philosophy



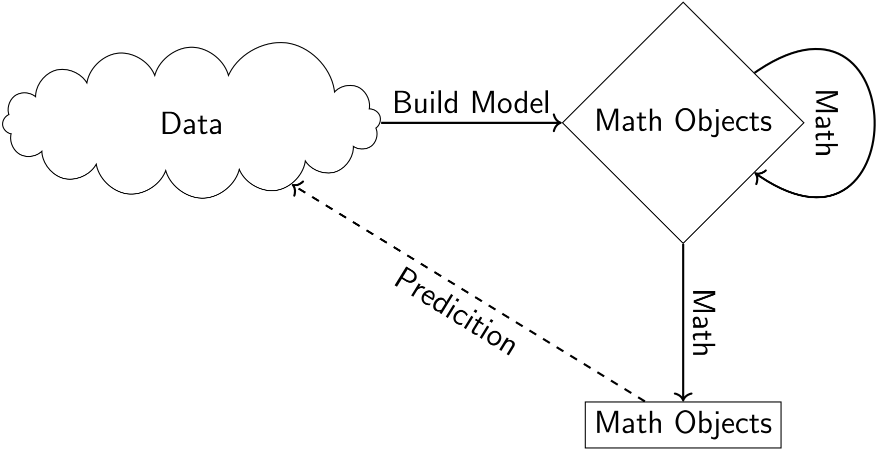
Philosophy



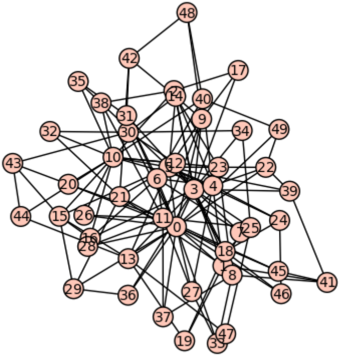
Philosophy



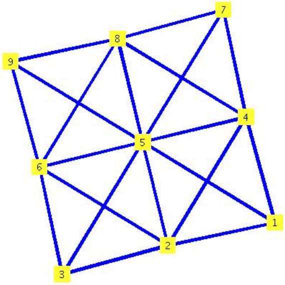
Philosophy


Data
Build Model
Math Objects
Math
Math
Predicition
Math Objects

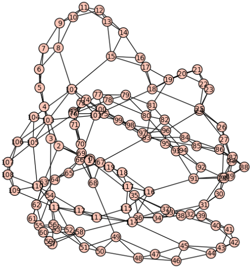
Complex Networks



Examples

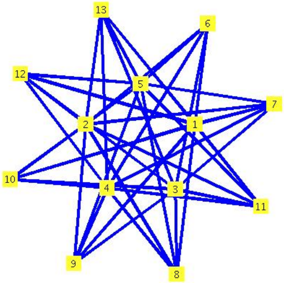


(a) Graph

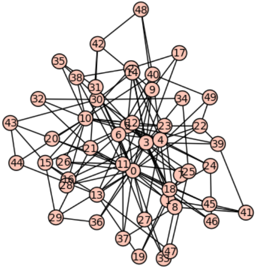


(b) Network

Examples

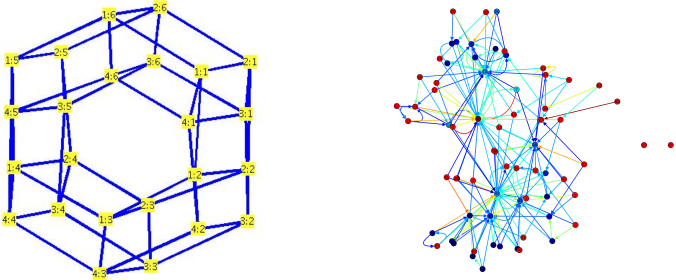


(a) Graph



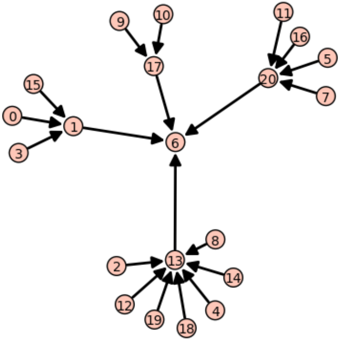
(b) Network

Examples

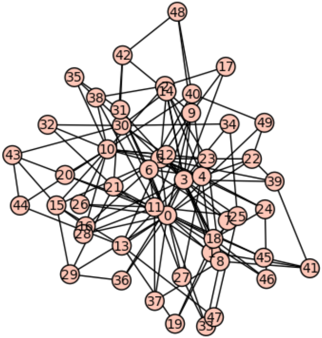


(a) Graph (b) Network

Centrality



Centrality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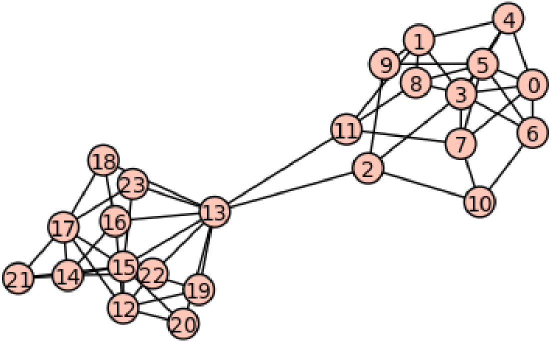
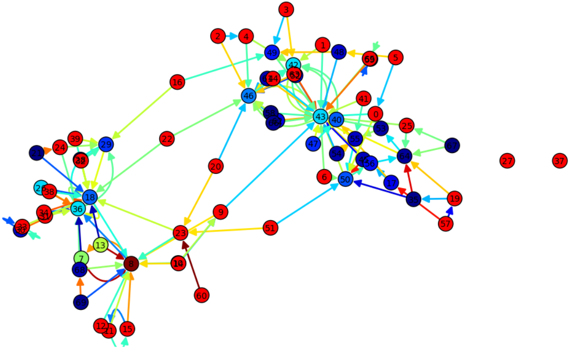
Clustering

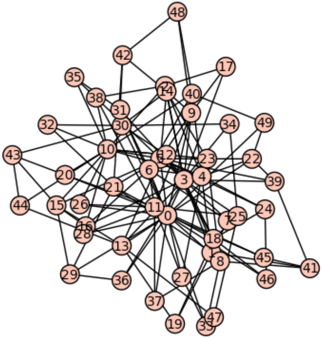


Dartmouth

Clustering



Clustering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11
0
25
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8
41
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19
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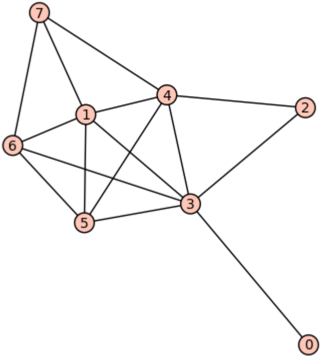
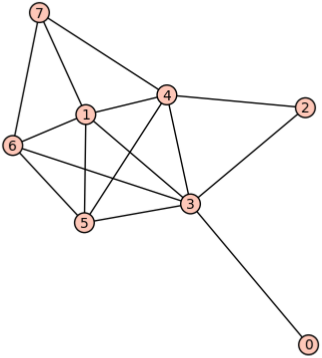
Degree Matrix



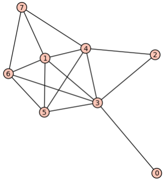
$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

Adjacency Matrix



$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Laplacian Matrix



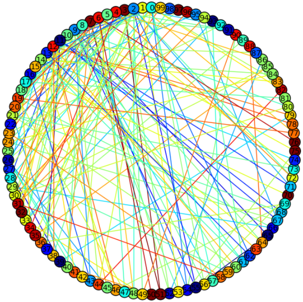
$$L = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 6 & -1 & -1 & -1 & 0 \\ 0 & -1 & -1 & -1 & 5 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & -1 & 4 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & -1 & 0 & 0 & -1 & 0 & -1 & 3 \end{bmatrix}$$

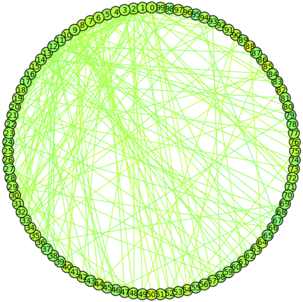
Spectral Graph Theory

Fan Chung: *Spectral Graph Theory*, AMS, (1997).

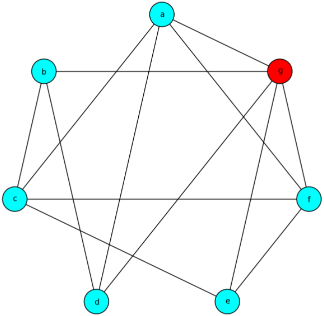
"Roughly speaking, half of the main problems of spectral theory lie in deriving bounds on the distributions of eigenvalues. The other half concern the impact and consequences of the eigenvalue bounds as well as their applications."

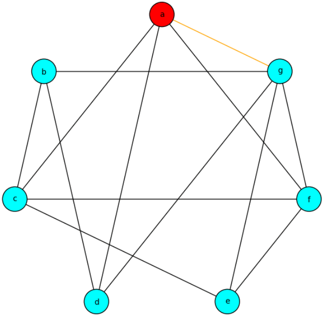
Diffusion





Random Walks





What is a multiplex?

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Definition

A *multiplex* is a collection of graphs all defined on the same node set.

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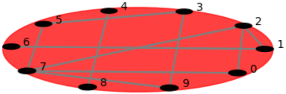
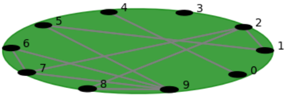
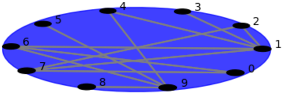


(a) Family



(b) Colleagues



(c) Facebook

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Definition

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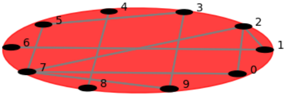
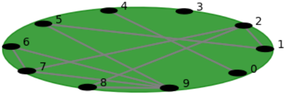
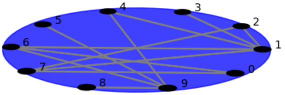


(a) Buses



(b) Trains



(c) Airplanes

World Trade Web[1]

Figure: World trade networks

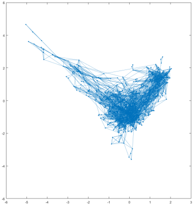
[1] R. Feenstra, R. Lipsey, H. Deng, A.C. Ma, and H. Mo. World Trade Flows: 1962-2000. NBER Working Paper 11040, (2005).

WTW Layers

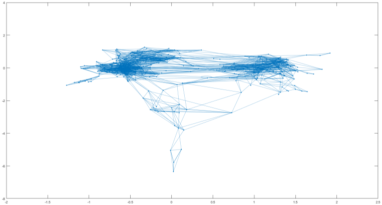
Layer	Description	Volume	% Total	Transitivity
0	Food and live animals	291554437	5.1	.82
1	Beverages and tobacco	48046852	0.9	.67
2	Crude materials	188946835	3.3	.79
3	Mineral fuels	565811660	10.0	.62
4	Animal and vegetable oils	14578671	0.3	.64
5	Chemicals	535703156	9.5	.83
6	Manufactured Goods	790582194	13.9	.87
7	Machinery	2387828874	42.1	.85
8	Miscellaneous manufacturing	736642890	13.0	.83
9	Other commodities	107685024	1.9	.56
All	Aggregate Trade	5667380593	100	.93

Table: Layer information for the 2000 World Trade Web.

Karnataka Village Data[1]



(a) Village 5



(b) Village 61

Figure: Karnataka Villages

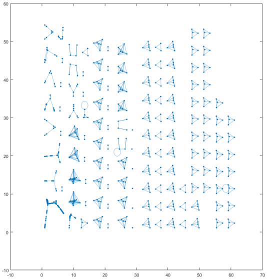
[1] A. Banerjee, A.G. Chandrasekhar, E. Duflo, and M.O. Jackson, The Diffusion of Microfinance. Science, (2013).

Village Layers

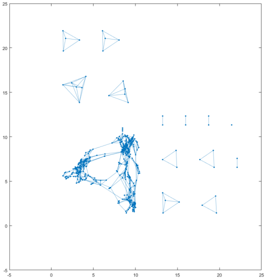
Layer	Village 4			Village 61		
Description	Density	Comp.	Giant %	Density	Comp.	Giant %
Borrow Money	.0082	26	.8354	.0108	15	.9188
Give Advice	.0077	49	.5892	.0098	34	.7377
Help Make Decisions	.0076	61	.1277	.0100	24	.8562
Borrow Kerosene or Rice	.0085	21	.8338	.0113	14	.9171
Lend Kerosene or Rice	.0086	22	.8308	.0113	14	.9255
Lend Money	.0081	14	.7908	.0107	17	.9036
Medical Advice	.0075	84	.2938	.0106	14	.9306
Friends	.0089	15	.9277	.0105	22	.8714
Relatives	.0085	29	.7231	.0105	26	.5448
Attend Temple With	.0073	117	.0462	.0089	108	.0372
Visit Their Home	.0087	15	.9185	.0116	11	.9475
Visit Your Home	.0088	16	.9108	.0117	11	.9492
Aggregate	.0121	3	.9862	.0155	8	.9679

Table: Layer information for two of the Karnataka Villages.

Medical Advice

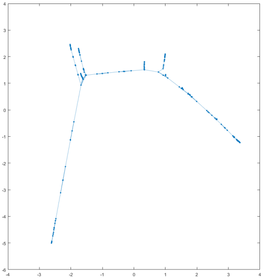


(a) Village 5

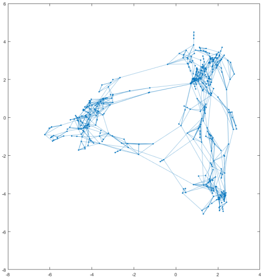


(b) Village 61

Medical Advice



(a) Village 5



(b) Village 61

Disjoint Layers

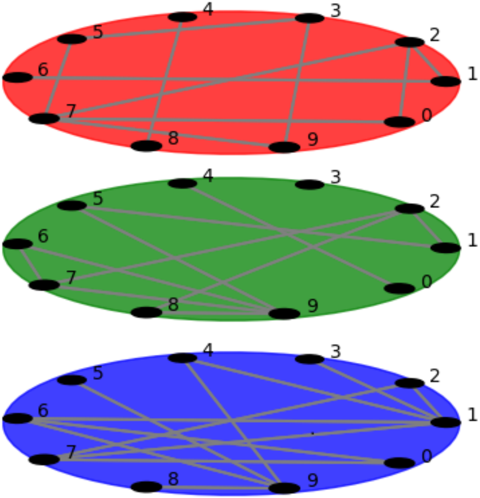
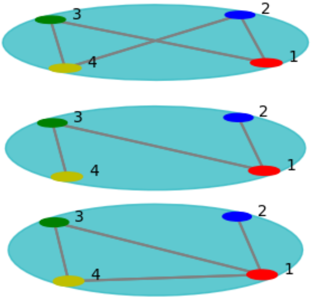
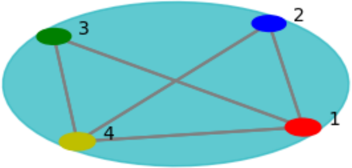


Figure: Disjoint Layers

Aggregate Models

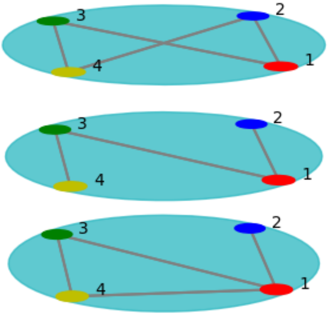


(a) Disjoint Layers

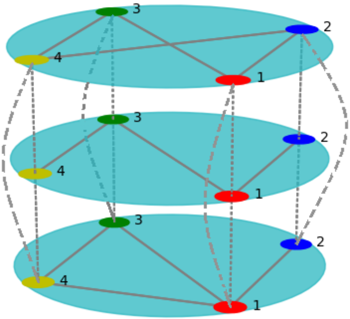


(b) Aggregate

Matched Sum



(a) Disjoint Layers



(b) Matched Sum

Algebraic Structure

We can represent the matched sum with a supra–adjacency matrix:

$$\begin{bmatrix} A^1 & wI_n & \cdots & wI_n & wI_n \\ wI_n & A^2 & \cdots & wI_n & wI_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ wI_n & wI_n & \cdots & A^{k-1} & wI_n \\ wI_n & wI_n & \cdots & wI_n & A^k \end{bmatrix}$$

where the A^α are the adjacency matrices of the individual layers and w is a connection strength parameter.

Structural Asymptotics

As the number of layers grows, what happens to the:

- Density?
- Degree Distribution?
- Transitivity?
- Average Path Length?
- Diameter?
- Clique Number?
- ...

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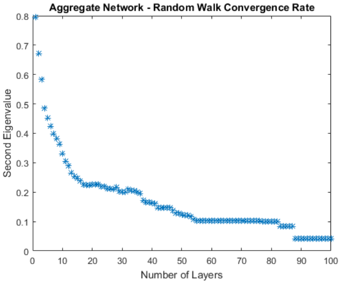
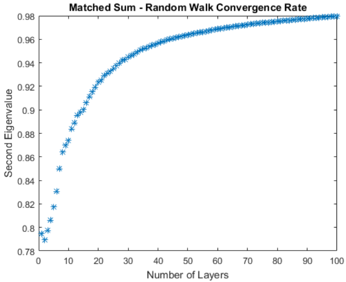
- ...
- **Dynamics!**

Random Walk Convergence



(a) Aggregate



(b) Matched Sum

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- Two types of interactions
  - Within the individual layers
  - Between the layers
- Effects should “pass through” nodes
- Two step iterative model

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$$(v')_i^\alpha = \sum_{\beta=1}^{k} m_i^{\alpha,\beta} c_i^{\alpha,\beta} (Dv)_i^\beta$$

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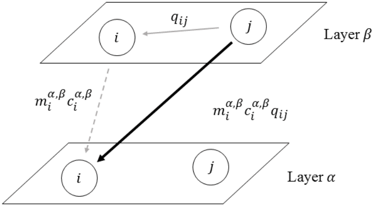


Matrix Realization

The matrix associated to the total operator also takes a convenient block form:

$$\begin{bmatrix} C_1D_1 & C_1D_2 & \cdots & C_1D_k \\ C_2D_1 & C_2D_2 & \cdots & C_2D_k \\ \vdots & \vdots & \vdots & \vdots \\ C_kD_1 & C_kD_2 & \cdots & C_kD_k \end{bmatrix}$$

Where the $\{D_i\}$ are the dynamical operators associated to the layers and the $\{C_i\}$ are the diagonal proportionality matrices.

Preserved Properties

If the dynamics on each layer are assumed to have certain properties, we can prove that those properties are preserved in our operator:

- Stochasticity
- Irreducibility
- Primitivity
- Positive (negative) (semi)–definiteness

Multiplex Random Walks

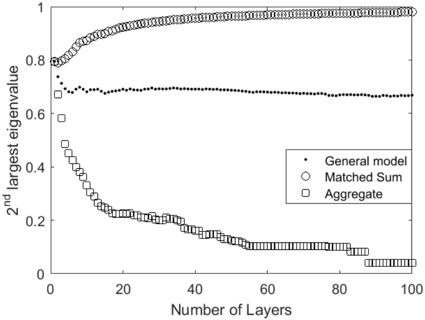


Figure: Comparison of random walk convergence for multiplex models.

D. DeFord and S. Pauls: A new Framework for Dynamical Models on Multiplex Networks, Journal of Complex Networks, (2017).

Laplacian Dynamics

Under our dynamical model, where effects pass through node copies to other layers, the heat diffusion interpretation of the Laplacian can be derived from first principles:

$$\frac{dv_i^\alpha}{dt} = -K \sum_{\beta=1}^{k} c_i^{\alpha,\beta} \sum_{n_i^\beta \sim n_j^\beta} (v_i^\beta - v_j^\beta).$$

$$\frac{dv_i^\alpha}{dt} = -K \sum_{\beta=1}^{k} c_i^{\alpha,\beta} (Lv)_i^\beta,$$

Laplacian Eigenvalue Bounds

Let $\{\lambda_i\}$ be the eigenvalues of $\mathscr{D}$ and $\{\lambda_i^\alpha\}$ be the eigenvalues of the α–layer Laplacian D^α. We have the following bounds:

- Fiedler Value:

$$\max_\alpha(\lambda_F^\alpha) \leq k\lambda_F \leq \lambda_F^m + \sum_{\beta \neq m} \lambda_1^\beta$$

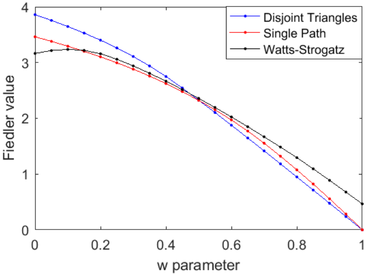
- Leading Value:

$$\max_i(\lambda_1^i) \leq k\lambda_1 \leq \sum_i \lambda_1^i$$

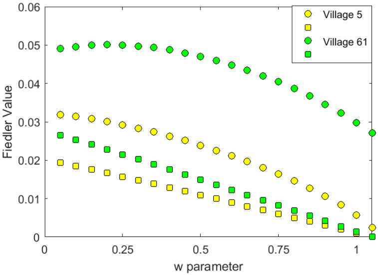
- General Form:

$$\max_i(\lambda_{n-j}^i) \leq k\lambda_{n-j} \leq \min_{J \vdash n+k-(j+1)} \left(\min_{\sigma \in S_n} \left(\sum_{\alpha=1}^{k} \lambda_{j_\alpha}^{\sigma(\alpha)} \right) \right)$$

Social Diffusion



(a) Synthetic



(b) Villages

Centrality Comparison

Figure: Comparison of multiplex eigenvector centrality scores. Varying the weighting scheme allows us to control how much mixing of centrality occurs between layers, while the matched sum model is just a linear transformation of the original rankings.

Centrality Comparison

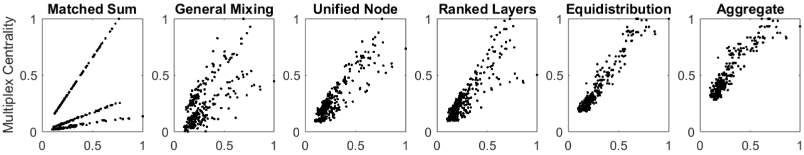
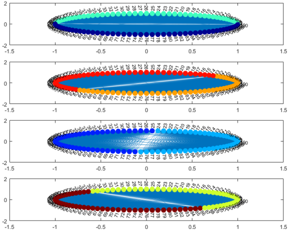
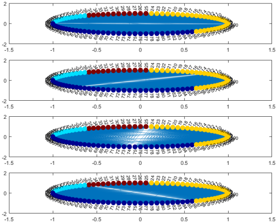


Figure: Comparison of multiplex eigenvector centrality scores. Varying the weighting scheme allows us to control how much mixing of centrality occurs between layers, while the matched sum model is just a linear transformation of the original rankings.

Clustering Comparison

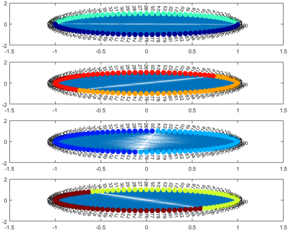


(a) Matched Sum



(c) Aggregate

Clustering Comparison



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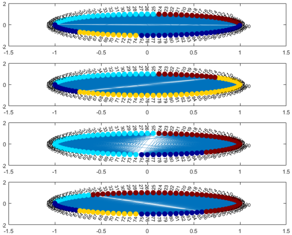
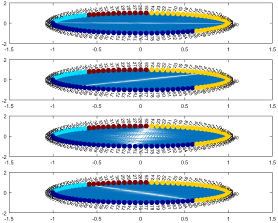


(b) Dynamical Model



(c) Aggregate

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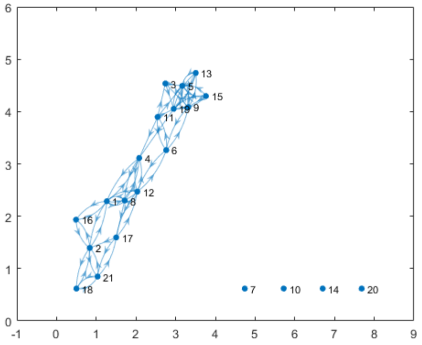
That's all...

Thank You!

Different Perspectives on “Friendship”



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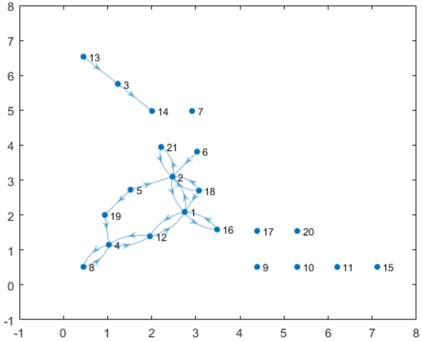

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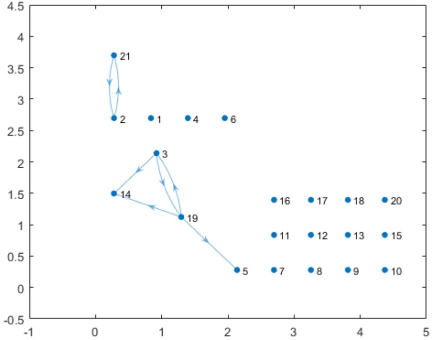

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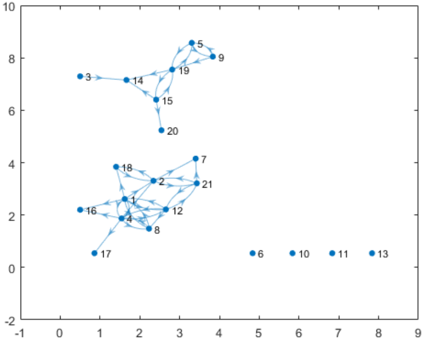

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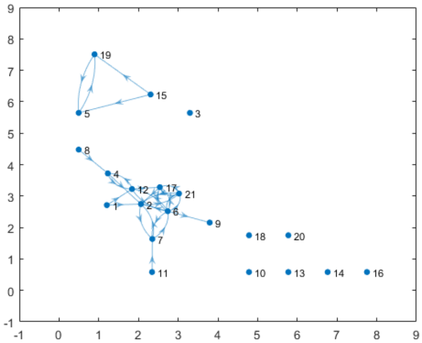

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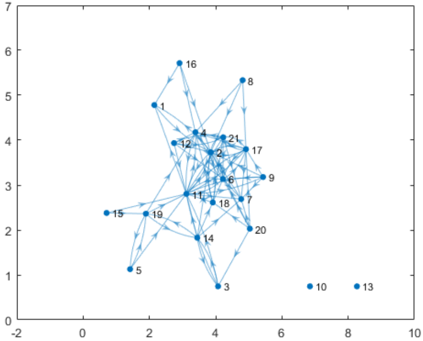

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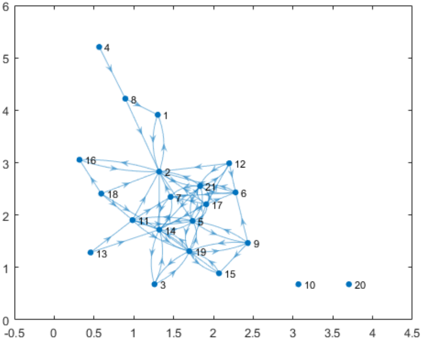

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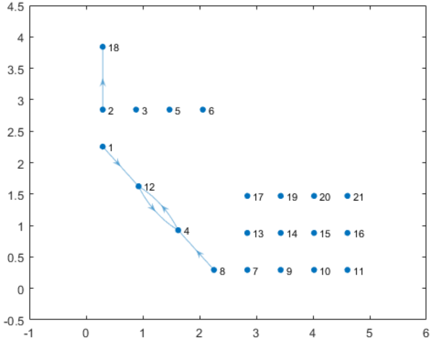

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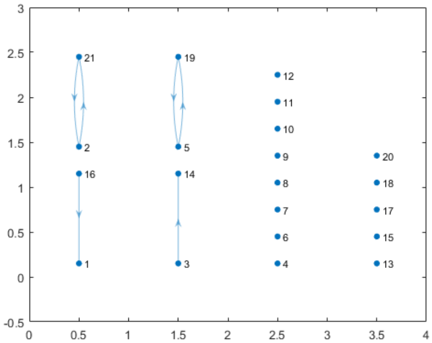

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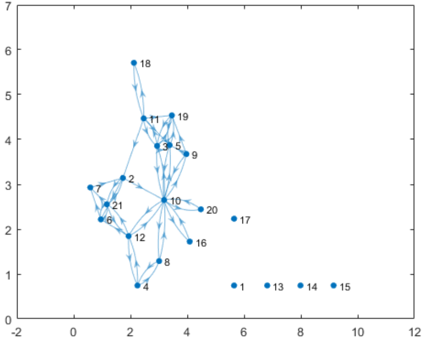

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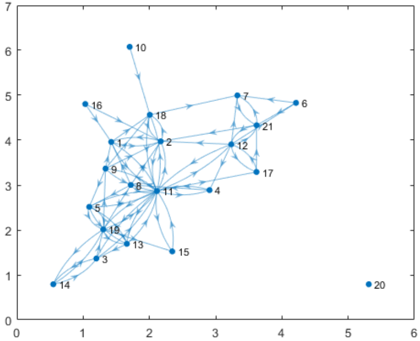

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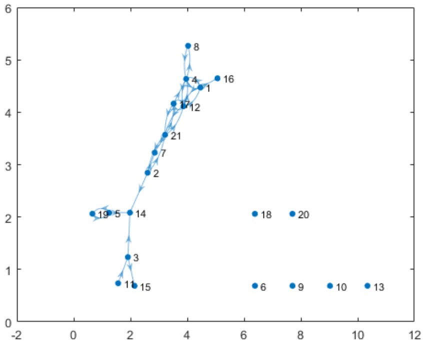

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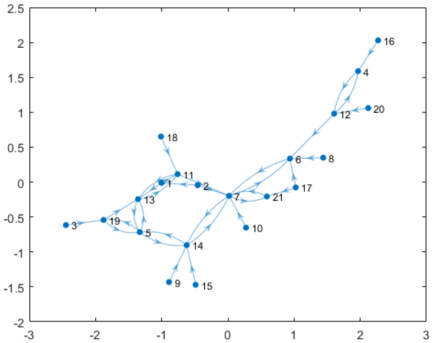

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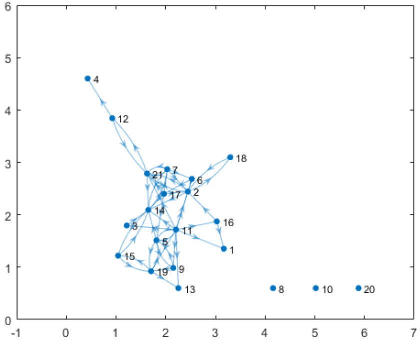

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Dartmouth

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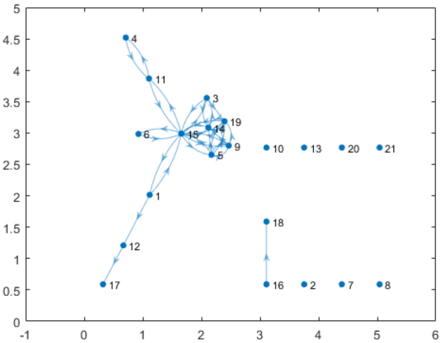

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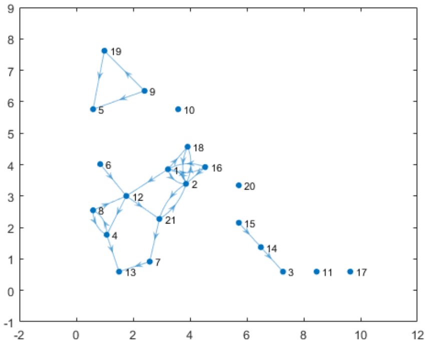

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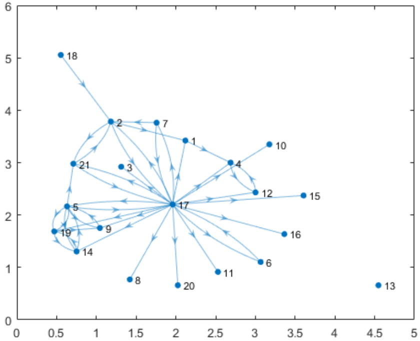

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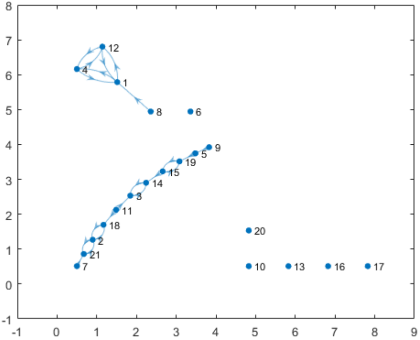

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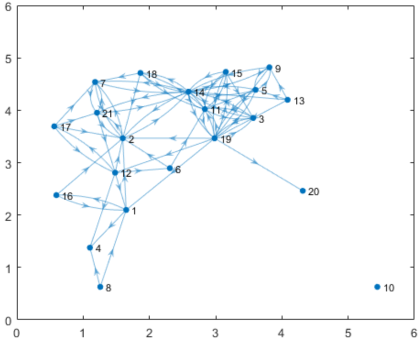

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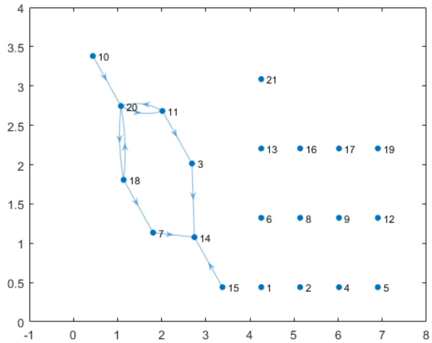

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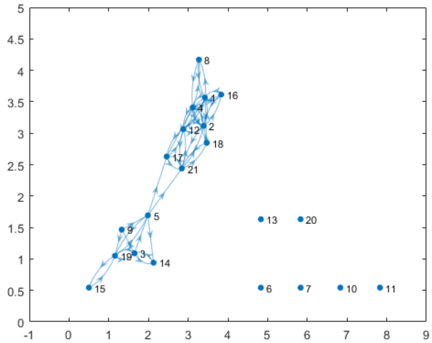

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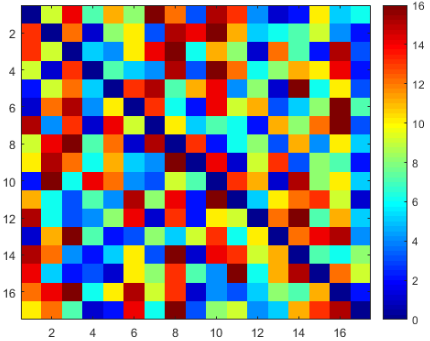
“Friendship” over Time



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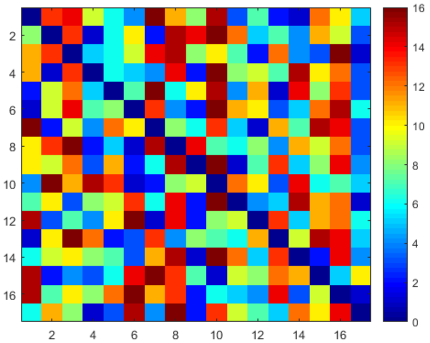

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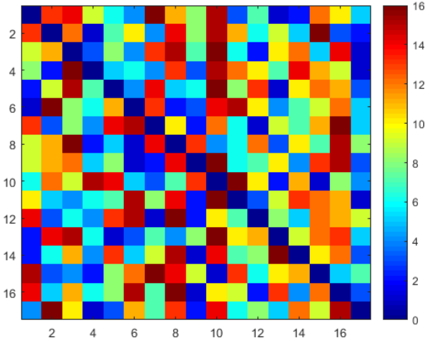

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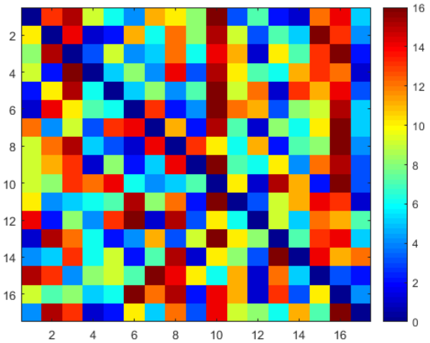

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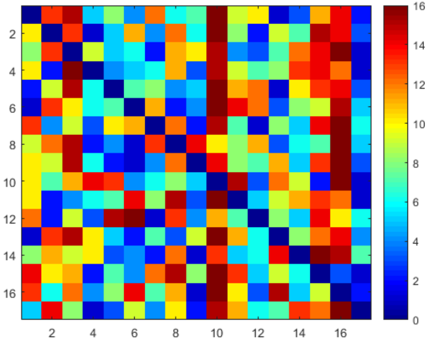

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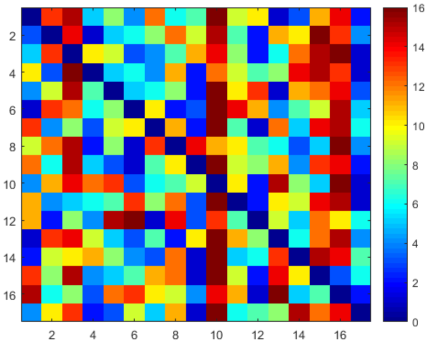

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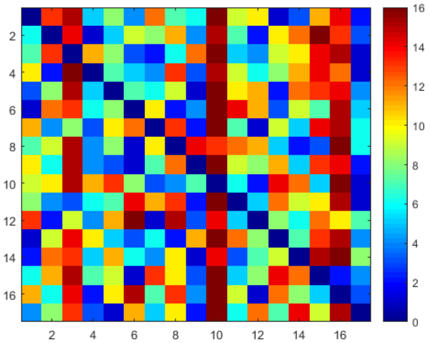

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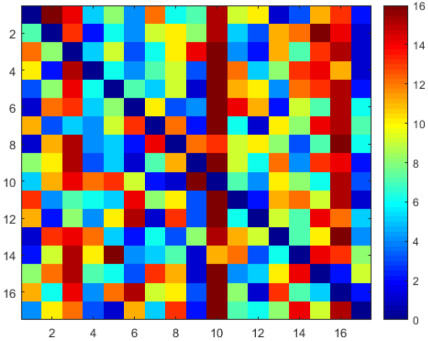

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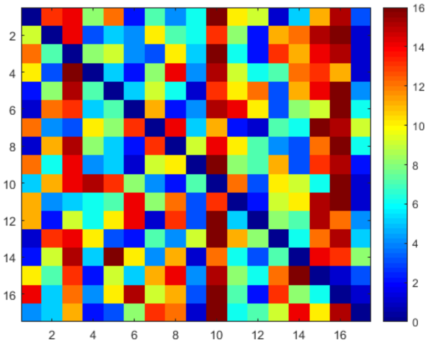

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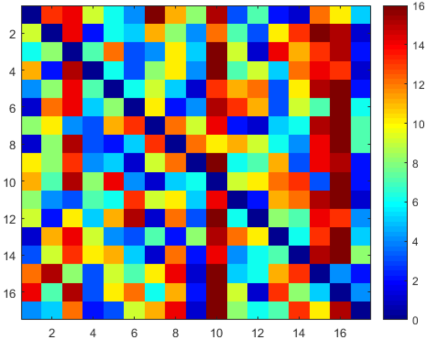

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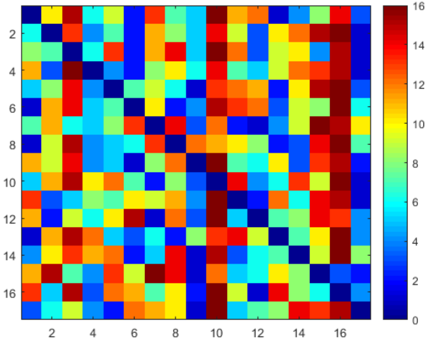

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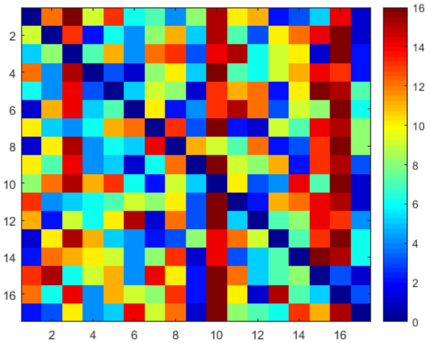

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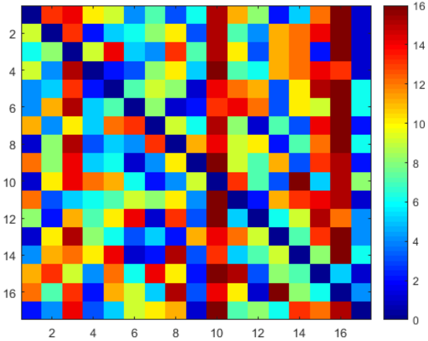

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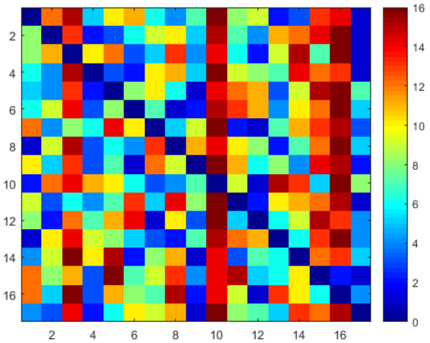

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Solution

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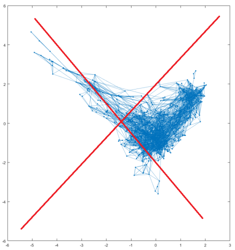

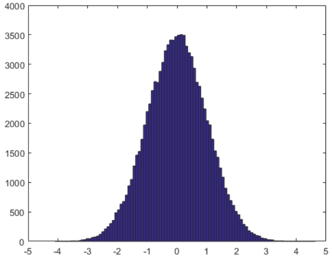

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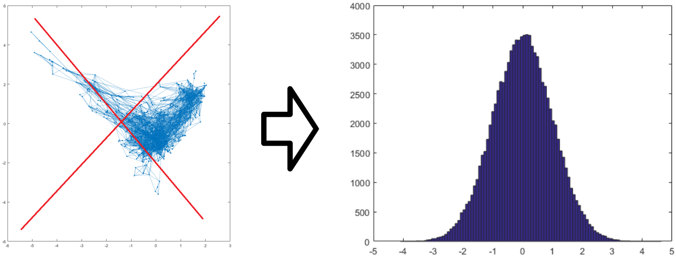

Definition (Null Model)

A random network, parameterized to match some features of a given network, used to compare “expected” network measures.

Erdos–Renyi

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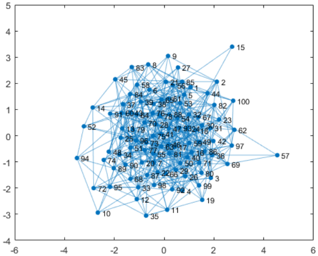
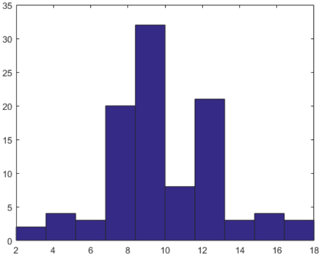


(f) Graph Example



(g) Degree Distribution

Barabasi–Albert (Centrality)

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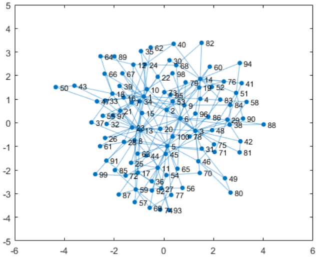
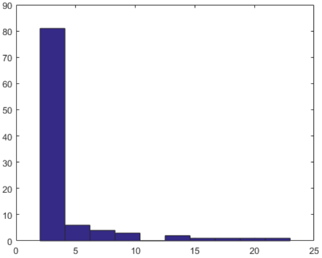


(j) Graph Example



(k) Degree Distribution

Watts–Strogatz (Local Clustering)

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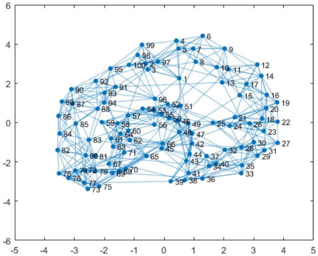
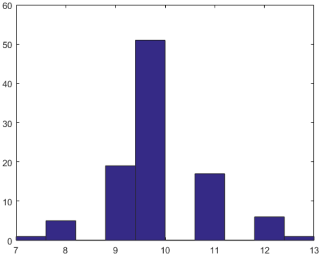


(n) Graph Example



(o) Degree Distribution

Stochastic Block Model (Global Clustering)

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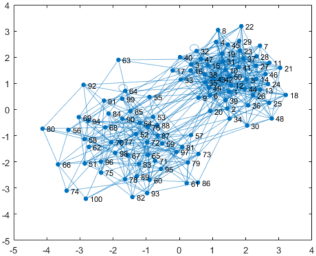
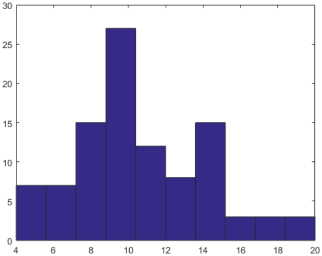


(r) Graph Example



(s) Degree Distribution

RDPM

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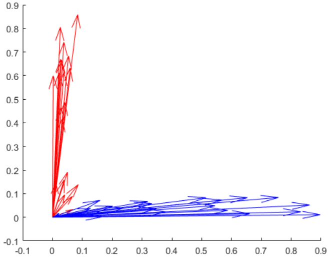
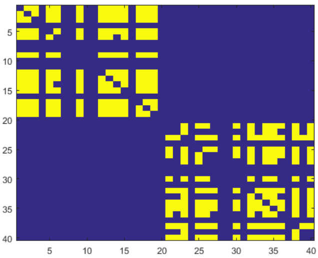
- Angle – Community assignment
- Magnitude – Centrality

Angle – Community Assignment

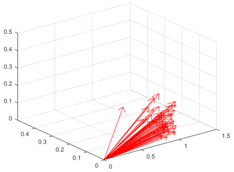


(t) Vectors

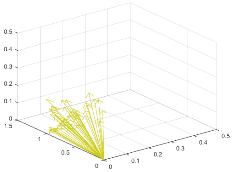


(u) Graph

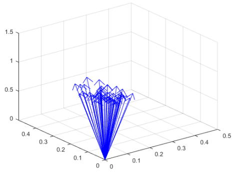
Example: Uniform Noise



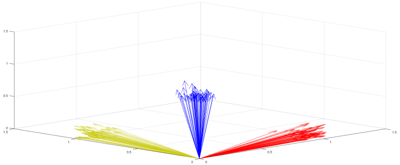
(v) Community 1 Vectors



(w) Community 2 Vectors



(x) Community 3 Vectors

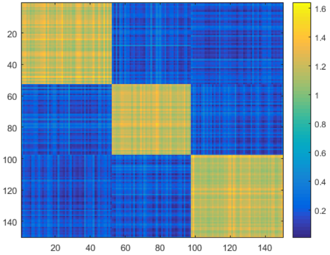


(y) All Vectors

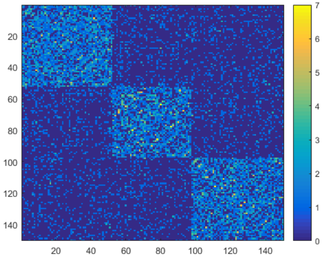
Figure

Dartmouth

Example: Uniform Noise

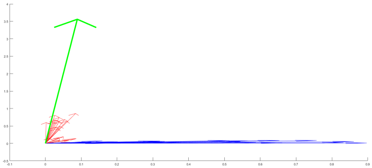


(a) Dot Products

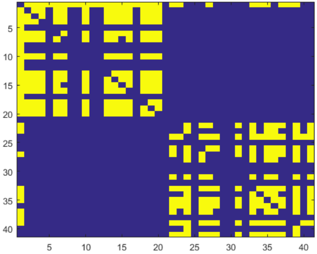


(b) WRDPM Network

Magnitude – Centrality

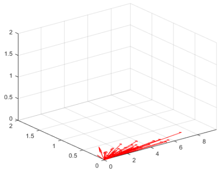


(c) Vectors

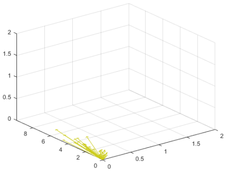


(d) Graph

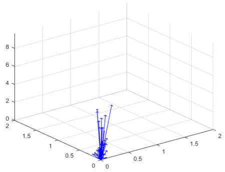
Example: Multiresolution Communities



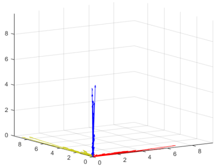
(e) Community 1 Vectors



(f) Community 2 Vectors

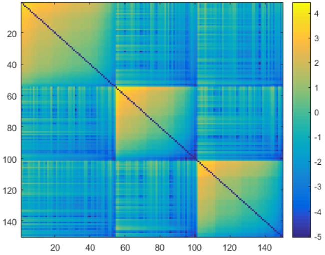


(g) Community 3 Vectors

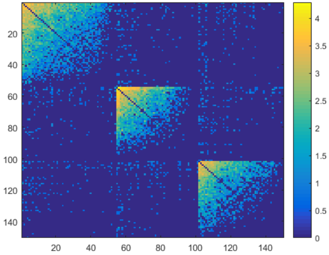


(h) All Vectors

Example: Multiresolution Communities



(i) Dot Products



(j) WRDPM Network

Edge Parameterized Models

Theorem

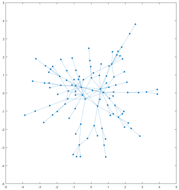
Let n be a fixed positive integer. For each pair (i, j) with $1 \leq i < j \leq n$ let $a_{i,j} = a_{j,i} \in \mathbb{R}$. Then there exist n real numbers $a_{\ell,\ell}$ for $1 \leq \ell \leq n$ such that the matrix $A_{i,j} = a_{i,j}$ is positive definite.

Corollary

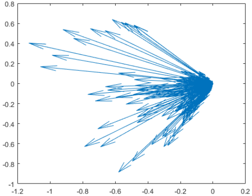
Any generative network model, on a fixed number of nodes n, where the edge weight between each pair of nodes is drawn independently from a fixed probability distribution, possibly with different parameters for each pair, can be realized under the WRDPM.

D. DeFord and D. Rockmore, A Random Dot Product Model for Weighted Networks, with D. Rockmore, arXiv:1611.02530, (2016).

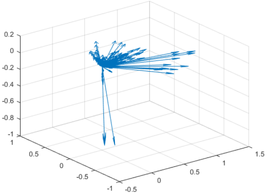
Unweighted Collaboration Network



(k) Collaboration Network



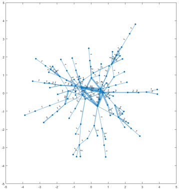
(l) Unweighted 2–Embedding



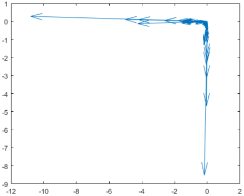
(m) Unweighted 3–Embedding

V. Batagelj and A. Mrvar: *Pajek datasets*, (2006).

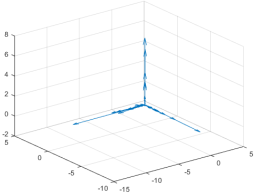
Weighted Collaboration Network



(n) Collaboration Network



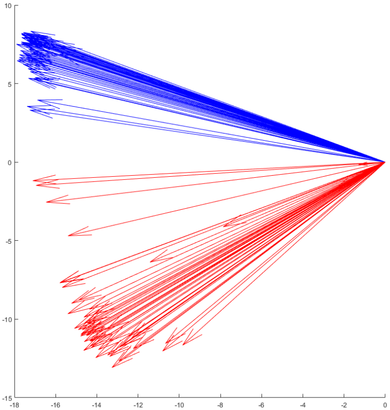
(o) Weighted 2–Embedding



(p) Weighted 3–Embedding

V. Batagelj and A. Mrvar: *Pajek datasets*, (2006).

Voting Data



J. Lewis and K. Poole: *Roll Call Data*,

voteview.com/dwnl.html.

That's all...

Thank You!

Dimension Selection

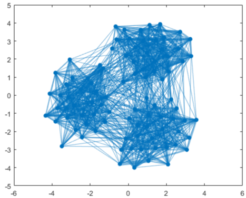
Since the dimension of the embedding is intrinsically related to the realized community structure it is natural to try and make use of this relationship to determine the right choice of d. Motivated by the case of disjoint communities, where if we have an effective, normalized embedding we should have

$$\langle X_i, X_j \rangle = \begin{cases} 1 & \text{i and j belong to the same community} \\ 0 & \text{i and j belong to different communities} \end{cases}$$

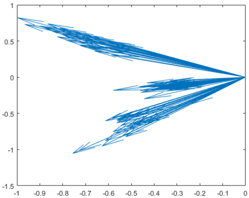
Thus, the sum of intra–community dot products should be $\sum_{i=1}^{\ell} \binom{z_\ell}{2}$. Similarly, the sum of the inter–community dot products should be 0. we define a stress function s depending on the community assignments after embedding.

$$s(d) = \sum_{i=1}^{d} \binom{z_i}{2} - \mathrm{s}_{\text{intra}}(d) + \mathrm{s}_{\text{inter}}(d)$$

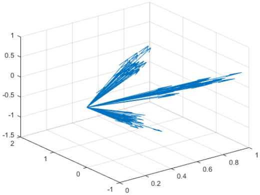
Dimension Example



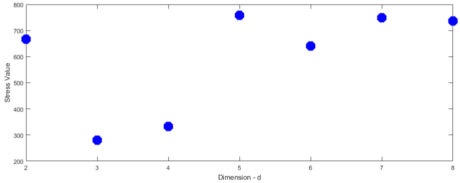
(q) Weighted Network



(r) 2-Embedding



(s) 3-Embedding



(t) Stress Function

Coauthorship Revisited

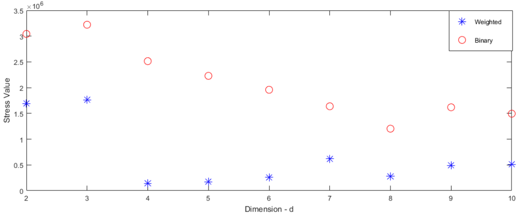


Figure: Comparison of stress values for the computational geometry coauthorship network between the weighted and unweighted realizations. The weighted embedding significantly outperforms the binarized model.