Combinatorial Rearrangements, Restricted Permutations, and Matrix Permanents

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Abstract

Enumerative Combinatorics is the study of counting problems. Frequently, we can use concrete models to reduce abstract problems to more manageable cases; an important example is the Fibonacci numbers, which can be described as the number of ways to tile a $1 \times n$ rectangle with squares and dominoes. We introduce a model based on seating rearrangements in a classroom and show how it can be applied to a wide variety of problems. Our main goal is to use the model to construct recurrences and closed form expressions for the problems that we consider. Using techniques from Combinatorics, Graph Theory, and Linear Algebra we will also prove several theorems about the combinatorial structures generated by our model.

Notation

We will use the notation $R(*)$ to represent the number of permissible rearrangements of a particular context-dependent structure. Subscripts will be used to identify classes of rearrangements.

Introduction

In the early 1990's, mathematicians from UCM calculated the number of possible rearrangements of the students in a rectangular classroom, given that each student could only move one desk in any direction [3, 7]. They solved the $2 \times n$ case by constructing a matrix to represent the number of rearrangements of a classroom of length $n+1$, and used mathematical induction to show that the total number is:

$$R(2,n) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} = F_{n+1}^2$$

In order to solve the general $2m \times n$ case they computed the permanent of a symbolic imaginary matrix to obtain the solution $R(2m, n)=$

$$2^{2mn} \prod_{t=1}^{2m} \prod_{s=1}^{n} \left(\cos^2\left(\frac{s\pi}{n+1}\right) + \cos^2\left(\frac{t\pi}{2m+1}\right) \right)$$

Seating Rearrangements

We began our research by constructing combinatorial proofs of the enumerations given in [3, 7]. To simplify the $2 \times n$ case, we colored the classroom like a chessboard, and considered the movements of the students from the white and black desks separately. We then constructed a bijection between the "colored" seating rearrangements and the Fibonacci tilings of a $1 \times n$ rectangle to show that $R(2,n) = F_{n+1}^2$. The characteristic polynomial of the recurrence, $\{-1, 1, \bar{\phi}^2, \phi\}$, lead us to an equivalent generalized power sum.

A similar technique allows us to reduce the general case as well. Coloring a $2m \times n$ classroom allows us to represent the movement of each student by a domino, covering the students original and final desks. This bijection leads to the following theorem:

Theorem 1. *The number of rearrangements of a $2m \times n$ classroom, $R(2m, n)$, is equal to the square of the number of domino tilings of a $2m \times n$ rectangle.*

This is a much simpler problem, as domino tilings were completely solved in the 1960's by Kasteleyn, Temperley, and Fisher [2]. An example of both the Fibonacci tilings and the general tilings can be seen in (fig. 1).

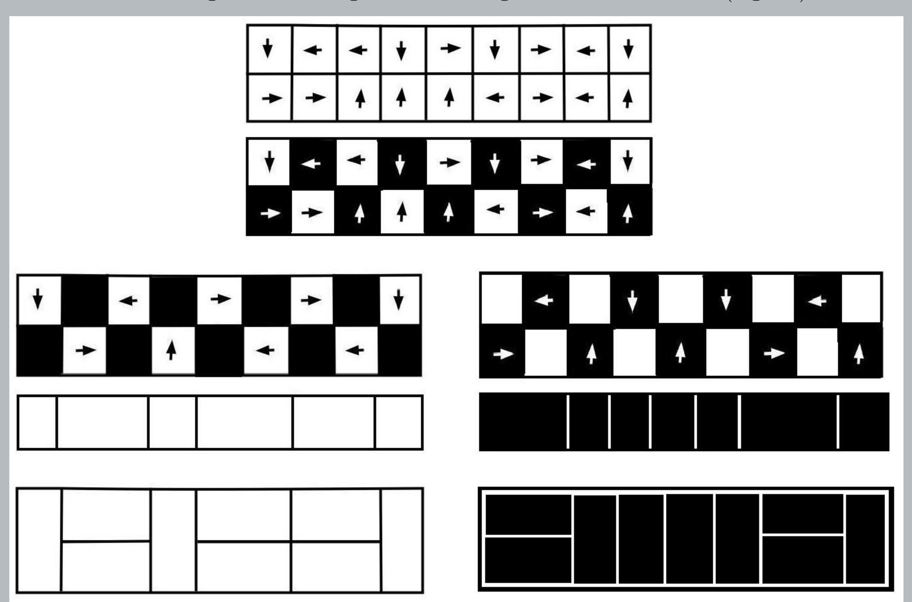


Figure 1: Domino Tilings Generated by a 2×9 Seating Rearrangement

3-D Prisms

Our first extension of this model dealt with the related question, "How many rearrangements exist in a $m \times n$ classroom if the students are permitted to remain in their seats?" Classifying the endings of a $2 \times n + 1$ classroom leads to the following homogeneous recurrence:

$$a_{n+1} = 2a_n + 5a_{n-1} + 4\sum_{i=3}^{n} a_{n-i} = a_{n+1} = 2a_n + a_{n-1} + 4\sum_{i=2}^{n} a_{n-i}$$

Then, computing $a_{n+2} - a_{n+1}$ gives $a_{n+1} = 3a_n + 3a_{n-1} - a_{n-2}$, and eigenvalues $\{-1, 2+\sqrt{3}, 2-\sqrt{3}\}$. Combined with the initial conditions $a_0 = 1$, $a_1 = 2$, and $a_2 = 9$ we constructed the generalized power sum

$$a_n = \left(\frac{1}{3}\right)(-1)^n + \left(\frac{2+\sqrt{3}}{6}\right)\left(2+\sqrt{3}\right) + \left(\frac{2-\sqrt{3}}{6}\right)\left(2-\sqrt{3}\right)$$

This sequence is equivalent to A006253 in the OEIS [6]. A006253 is defined by the number of tilings of a $2 \times 2 \times n$ prism in $\mathbb{R}^3$ with $1 \times 1 \times 2$ dominoes. We proved that these two structures are equivalent by demonstrating a bijection between the seating rearrangements and the 3-D tilings. An example of the mapping can be seen in (fig. 2).

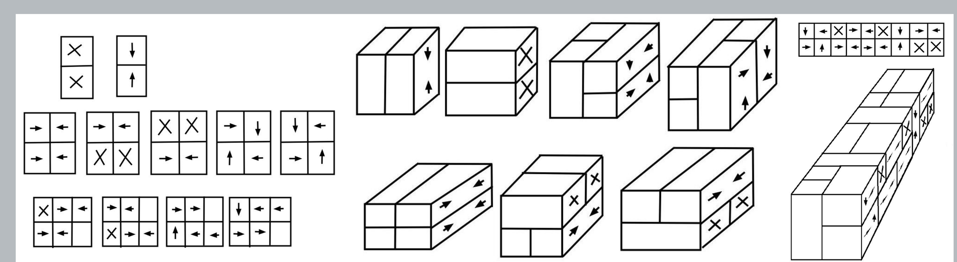
The $3 \times n$ case has significantly more endings to consider. We were able to exploit some of the symmetries of the structure to generate a ten term recurrence for $R_s(3, n)$:

$$a_{n+1} = 6a_n + 21a_{n-1} - 42a_{n-2} - 89a_{n-3} + 68a_{n-4}$$
$$+89a_{n-5} + 42a_{n-6} + 21a_{n-7} + 6a_{n-8} + a_{n-9}$$

Taken with the initial conditions given in (tab. 1) this sequence is equivalent to the number of perfect matchings of $P_2 \times P_3 \times P_n$ as computed by Lundow [4]. More generally we showed that this relation holds for any integer m.

Theorem 2. *The number of seating rearrangements, where students are allowed to remain in their seats, in an $m \times n$ classroom, $R_s(m, n)$, is equal to the number of perfect matchings in $P_2 \times P_m \times P_n$.*

Currently, matchings of this sort are calculated through extensive matrix manipulations and numerical algorithms. By using our model to express them as counting problems, we are able to generate recurrences and closed form expressions, which are much easier to implement.



$2 \times n$ Endings — $2 \times 2 \times n$ Endings — Bijection

Figure 2: $2 \times n$ Seating Rearrangements and $2 \times 2 \times n$ Prism Tilings

n	1	2	3	4	5	6	7	8	9
$R(3,n)$	3	32	229	1845	14320	112485	880163	6895792	54003765

Table 1: $R_s(3, n)$ initial conditions

Generalizations

Generalizing our model allows us to count in more abstract circumstances. This is an important example of the applicability of the model; by finding formal generalizations, we can reduce complex instances of problems in other fields to questions about seating rearrangements that we can answer combinatorially.

- **Set Theory.** Given a set $S = \{s_1, s_2, s_3, \dots s_k\}$ we define a restriction, $R = \{A_1, A_2, A_3 \dots A_k\}$, on S, where each $A_i \subseteq \mathcal{P}(S)$. Then, we want to count the number of permutations, $\sigma_R(S)$, that satisfy $\sigma(s_i) \in A_i$ for all $1 \leq i \leq k$. Note that there are $\left(k^2\right)^k$ possible R for any given set with k elements.
- **Digraphs.** Define a digraph $D = (V, E)$, where the vertices in V are the elements being rearranged, and for any $x, y \in V$, let $(x, y) \in E$ if an element in position x is permitted to move to position y. Then, we want to count the number of cycle covers of D. This is equivalent to computing the permanent of a $0-1$ matrix, which is difficult in general [8]. Permanents also have a deep connection to graph matchings, which have many applications. Thus, our model can be used to provide expressions for these difficult and important problems.

Fibonacci and Lucas Tilings

Two of the best known integer sequences are the Fibonacci sequence and the Lucas sequence. We can model both of these sequences with our model and its generalizations. The seating rearrangements and digraphs are straightforward constructions, but the set theory model is more complex. To count the Fibonacci numbers we let $S = \{1, 2, 3 \dots k\}$, and define each $A_i = \{i-1, i, i+1\}$. To define the Lucas numbers, perform the addition (mod k).

Permanents and Cycle Covers

Given a digraph, D, the adjacency matrix of D is a $0-1$ matrix whose entries represent the edges in the digraph. The permanent of a matrix is defined exactly like the determinant, without taking in to account the the sign of the permutation. Unfortunately, this small change means that there are no exact polynomial-time algorithms to calculate the permanent of an arbitrary matrix [8]. The best known algorithm to compute a permanent, and thus the quickest deterministic way to compute the cycle covers of a digraph, is Ryser's Algorithm.

Ryser's Algorithm. *Given a zero-one, $n \times n$ matrix M. Denote a sub-matrix of M created by deleting exactly r rows of M as M_r. Then let $RSP(M_r)$ represent the product of the row sums of M_r. Compute the sum of all possible $RSP(M_r)$ over each possible value of r. This gives us the final expression for the permanent of a matrix as*

$$per(M) = \sum_{r=0}^{n} \sum_{i=1}^{\binom{n}{r}} -1^r RSP(M_r)$$

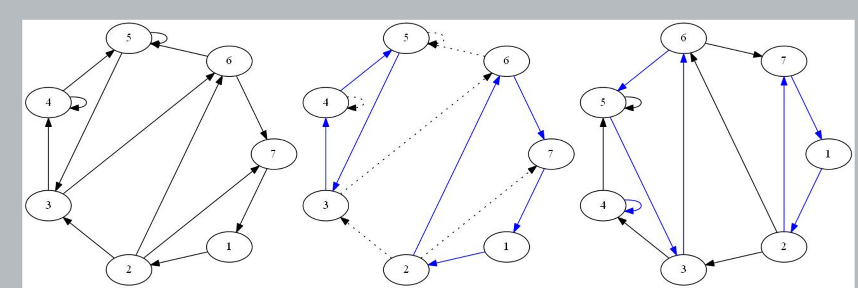
This algorithm is still $\mathbf{O}(2^n n^2)$ in complexity time, but that is much better than the $\mathbf{O}(n!)$ of cofactor expansion. Sometimes combination methods offer improvements for sparse graphs. Also, if the matrix is planar and bipartite, we may instead form a biadjacency matrix and use the pfaffian method or FTK Algorithm.

Consider the digraph shown in (fig. 3). Its adjacency matrix, A, is

$$A = \begin{bmatrix} 0&1&0&0&0&0&0 \\ 0&0&1&0&0&1&1 \\ 0&0&0&1&0&1&1 \\ 0&0&0&1&1&0&0 \\ 0&0&1&0&1&0&0 \\ 0&0&0&0&1&0&1 \\ 1&0&0&0&0&0&0 \end{bmatrix}$$

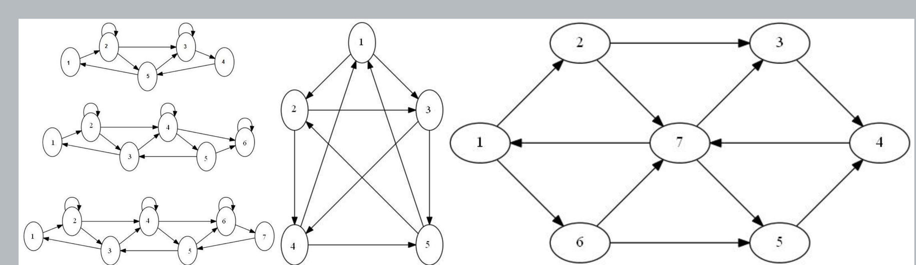
Since $Per(A) = 2$, we know that there are two cycle covers, which are also shown in (fig. 3). In 1966, Harary and Beineke showed that the adjacency matrix of a digraph has its permanent equal to its determinant if and only if the digraph contains no odd cycle covers [1]. A cycle cover is defined to be odd if it contains an odd number of even cycles, or even if it contains an even number of even cycles.

Thus, cycle cover 1 is an odd permutation because it is composed of one even cycle and one odd cycle. Cycle cover 2 is composed of three odd cycles and no even cycles, so it is an even cycle cover. Since determinants are much easier to compute than permanents, it is simpler to obtain closed forms for the number of rearrangements on a graph if we can compute the cycle covers in terms of determinants. To this end, we constructed several families of graphs with equal permanent and determinant (fig. 4).



Original Digraph — Cycle Cover 1 — Cycle Cover 2

Figure 3: Cycle Covers of a Digraph[1]



$G_{5,6,7}$ — K_5 — W_6

Figure 4: Digraphs with $Per(A) = Det(A)$[1]

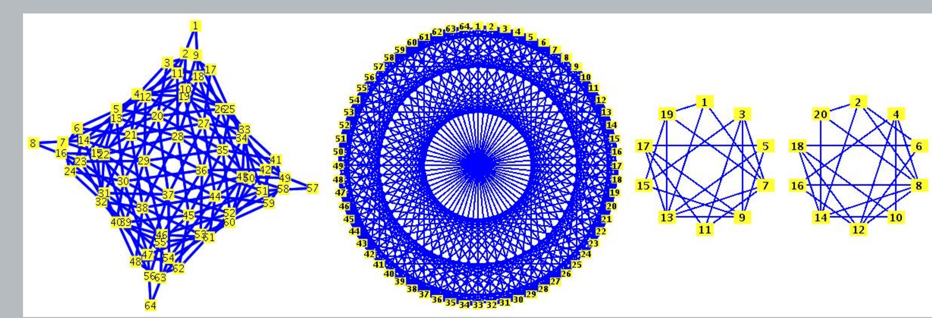
Chess Problems

As an extension of the original problem, we asked the following question, "Given a $m \times n$ chessboard, with a single game piece on each square, how many rearrangements are possible if each piece must make exactly one legal move?" At most one move? Can these problems be solved with recurrence techniques?

We began by considering rearrangements of kings, queens, bishops, knights, and rooks, on boards of size $1 \times n$, $2 \times n$, $3 \times n$, and $4 \times n$. Several of the sequences were equivalent to ones already in the OEIS, but many were not. The rearrangements for the kings, bishops, and knights tend to satisfy recurrences, while queen and rook rearrangements do not. However, the number of terms in the rearrangements grows geometrically. For example, the king rearrangements satisfy recurrences of order 2, 3, 10, 27, and 53, for boards with one through five rows respectively.

These chess problems can also be stated in terms of digraphs. Examples of these graphs are shown in (fig. 5). Notice that since the bishops must remain on a fixed color, their graph consists of two connected components, while the knights, who alternate between colors each move, have a bipartite graph. We counted the number of rearrangements for these structures by constructing the appropriate digraph and computing its permanent numerically. Similarly, we computed the rearrangements with stays permitted by adding the appropriate identity matrix to the adjacency matrix and recomputing the permanent.

A famous problem in graph theory and combinatorics is the Knight's Tour. This question asks for the number of distinct Hamiltonian cycles on the 8×8 knight graph. The number of tours was finally computed to be **26,534,728,821,064** in 1997 [5]. We asked a similar question about knight rearrangements. In order to determine the number of rearrangements on a 8×8 board we took advantage of the bipartition and constructed a biadjacency matrix. This left us a 32×32 matrix. Expanding the least dense rows by hand allowed us to sum over the permanents of 4,096 24×24 matrices leading to our final answer. The number of knight rearrangements on an 8×8 board is **8,121,130,233,753,702,400**.



Knight Moves — Rook Moves — Bishop Moves

Figure 5: Chessboard Rearrangement Graphs[2]

Checkers Problems

We also computed these values for checkers pieces. Checkers without stays only permit rearrangements when both m and n are even. Checkers rearrangements with stays lead to systems of linear recurrences, which can be solved combinatorially, without using permanents. These systems come from analyzing the endings based on the number of checkers in the last column. The following system represents the $2 \times n$ checkerboard rearrangements.

$$a_n = b_{n-1} + 2a_{n-2}$$

and

$$b_{n-1} = a_{n-2} + 2b_{n-2} + 4a_{n-4}$$

Substitution then gives a final solution

$$a_n = 2a_{n-1} + 3a_{n-2} - 4a_{n-3} + 4a_{n-4}$$

Fibonacci Relations

These chess piece rearrangements provided us with more interesting relations to the Fibonacci numbers.

- The number of $1 \times n$ king rearrangements is equal to the n^{th} Fibonacci number.
- The number of $2 \times n$ bishop rearrangements is equal to the square of the n^{th} Fibonacci number.
- The number of $2 \times 2n$ knight rearrangements is equal to the fourth power of the n^{th} Fibonacci number.
- The number of $2 \times 2n - 1$ knight rearrangements is equal to the product of the squares of the n^{th} and $n - 1^{st}$ Fibonacci numbers.

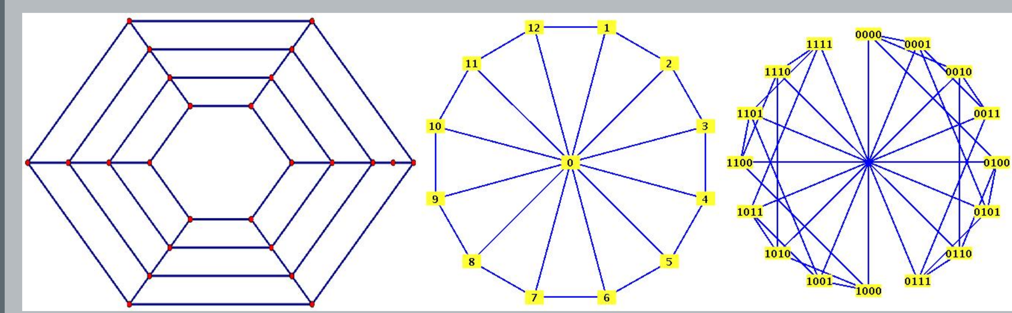
Graph Families

Finally, we applied the rearrangement model to three well-known families of graphs, in order to analyze their perfect matchings (fig. 6). Perfect matchings are often used by computational chemists to analyze organic compounds, they are also used by theoretical physicists to model atomic interactions [2]. These studies led to the following three theorems.

Theorem 3. *The number of rearrangements on the n^{th} iteration of a $k-$polygonal lattice, $R_l(k, n)$, is equal to*
- *the number of perfect matchings on $P_2 \times C_k \times P_n$ if k is even or*
- *the number of perfect matchings on $C_k \times P_n$ if k is odd.*

Theorem 4. *The number of rearrangements on the n^{th} wheel graph, $R_w(n)$ is equal to n^2.*

Theorem 5. *The number of rearrangements without stays on an $n-$cube, $R_H(n)$, is equal to the square of the number of rearrangements with stays on a $n-1$ cube, $R_{Hs}(n-1)$, which is in turn equal to the number of perfect matchings on an n cube.*



Polygonal Lattice — Wheel Graph — Hypercube

Figure 6: Arbitrary Graph Families[3,2,2]

Conclusions and Extensions

Our combinatorial seating rearrangement model can be applied to many subjects, both in mathematics and the physical sciences. Using combinatorial arguments to generate closed forms and recurrences is a valuable technique for describing the structure that underlies these problems. We also hope to be able to extend our results by considering the following problems:

- Adapting this model to analyze problems in crystal physics combinatorially
- Computing the remaining 8×8 chessboard rearrangements
- Finding a simple recurrence for the Wheel Graphs with stays
- Determining which "well-known" sequences can be motivated with our model
- Extending our model to Tori, Möbius Strips, the projective plane and other surfaces
- Constructing interesting families of digraphs with $Per(A) = Det(A)$

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