Enumerating Distinct Chessboard Tilings and Generalized Lucas Sequences Part I

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Roadmap

Today's Outline

Recurrence Relation Solutions

Problem

Recurrence Relation Solutions

Problem

High order recurrence relation solutions are unsatisfactory.

Chessboard Tiling Sequences

We are interested in integer sequences formed in the following fashion:

Definition

Given a fixed set of tiles T and integer k let T_n be defined as the number of ways to tile a $k \times n$ board with tiles in T as n ranges over the natural numbers.

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- Existence proof by constructing such a relation
- **Unfortunately...**

Upper Bound

The constructed relation is **spectacularly bad** and not feasible to compute for any interesting problem. For instance the following table shows the bound given in the proof and the actual minimal recurrence order for a simple family of tiling problems:

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Table: Simple Example

k	1	2	3	4	5	6	7	8	9	10
Upper Bound	1	4	9	25	64	169	441	1,156	3,025	7,921
$\mathcal{O}(T_n)$	1	2	2	3	4	6	8	14	19	32

Why it matters...

- Initial Conditions

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- Other Forms/Root Finding

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- Initial Conditions
- Other Forms/Root Finding
- Identities

Example Parameters

Example

Let T contain 1×1 and 2×2 squares and $T_{k,n}$ be the number of ways to tile a $k \times n$ rectangle with T.

Methods

There are many methods to approach these problems:

- Computationally
- Cayley–Hamilton Theorem
- Count indecomposable blocks
- ...

For theoretical results the successor operator method is very convenient.

Successor Operator Method

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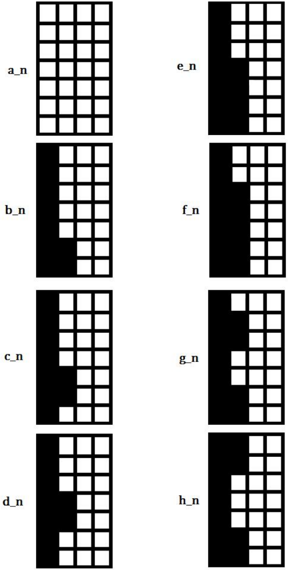
2. Re–write as a system of polynomials in the successor operator
3. Form a matrix from these polynomials
4. The symbolic determinant of this matrix is an annihilating polynomial for the sequence T_n

$7 \times n$ Rectangle Endings



$7 \times n$ Rectangles

$$
\begin{aligned}
a_n &= a_{n-1} + 5a_{n-2} + 2b_{n-1} + 2c_{n-1} + \\
&\quad 2d_{n-1} + 2e_{n-1} + 4f_{n-1} + 2g_{n-1} + h_{n-1} \\
b_n &= a_{n-1} + b_{n-1} + c_{n-1} + 2d_{n-1} + e_{n-1} + 2f_{n-1} \\
c_n &= a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1} + e_{n-1} \\
d_n &= a_{n-1} + 2b_{n-1} + c_{n-1} + g_{n-1} + h_{n-1} \\
e_n &= a_{n-1} + b_{n-1} + c_{n-1} \\
f_n &= a_{n-1} + b_{n-1} \\
g_n &= a_{n-1} + d_{n-1} \\
h_n &= a_{n-1} + 2d_{n-1}
\end{aligned}
$$

$7 \times n$ Rectangles

$$M = \begin{bmatrix} E^2 - E - 5 & -2E & -2E & -2E & -2E & -4E & -2E & -E \\ -1 & E-1 & -1 & -2 & -1 & -2 & 0 & 0 \\ -1 & -1 & E-1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -2 & -1 & E & 0 & 0 & -1 & -1 \\ -1 & -1 & -1 & 0 & E & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & E & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & E & 0 \\ -1 & 0 & 0 & -2 & 0 & 0 & 0 & E \end{bmatrix}$$

$7 \times n$ Rectangles

A Characteristic Polynomial:

$$det(M) = E^9 - 3E^8 - 30E^7 + 17E^6 + 138E^5 - 85E^4 - 116E^3 + 42E^2 + 32E$$

Recurrence Relation:

$$a_n = 3a_{n-1} + 30a_{n-2} - 17a_{n-3} - 138a_{n-4} + 85a_{n-5} + 116a_{n-6} - 42a_{n-7} - 32a_{n-8}$$

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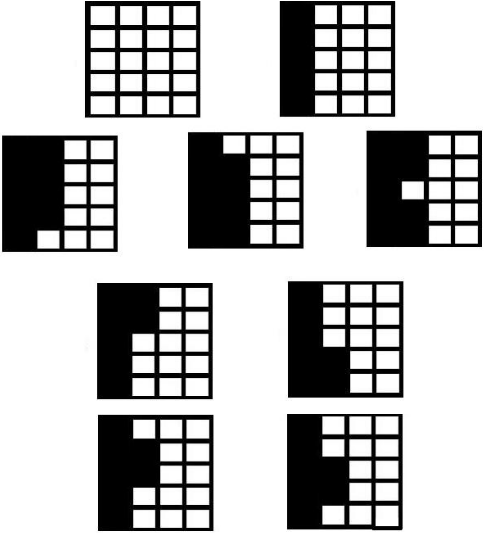
- Symmetry
- NOTE: No guarantee of minimality

$k = 5$



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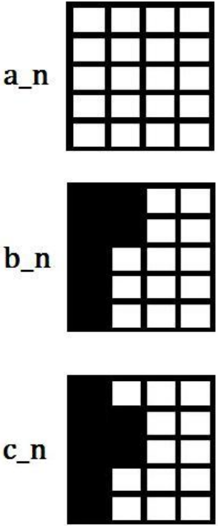
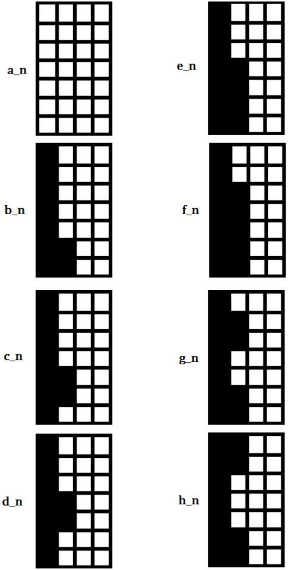


$k = 7$



Counting Distinct Tilings

- Burnside's Lemma

$$\mathcal{O}(T_{k,n}) = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|$$

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- Pòlya's Enumeration Theorem
- **Difficulties**

Example: Preliminaries

In order to prove the upper bounds for each sequence constructed by our example, we need some simple lemmas:

Example: Preliminaries

Lemma

The number of distinct Fibonacci tilings $\mathcal{S}(f_n)$ of order n up to symmetry is equal to $\frac{1}{2}(f_{2k} + f_{k+1})$ when $n = 2k$ and $\frac{1}{2}(f_{2k+1} + f_k)$ when $n = 2k+1$.

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Example: Preliminaries

Lemma

The number of distinct Padovan tilings $\mathcal{S}(P_n)$of order n up to symmetry is equal to $\frac{1}{2}(P_{2k} + P_{k+2})$ *when* $n = 2k$ *and* $\frac{1}{2}(P_{2k+1} + P_{k-1})$ *when* $n = 2k + 1$.

Example: Preliminaries

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The number of endings with no consecutive 1×1 *tiles is equal to* P_{n+2}.

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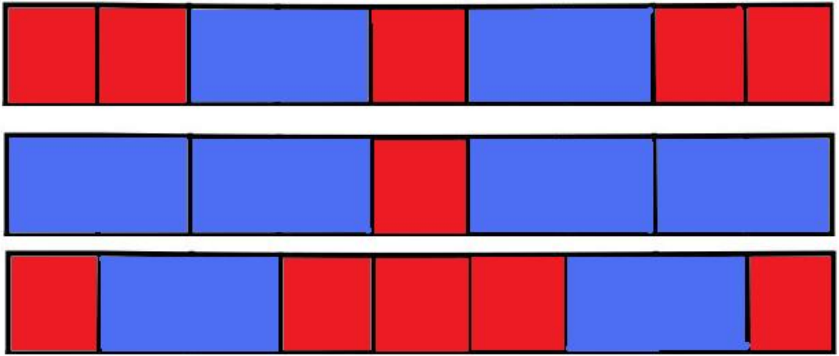
Lemma Proofs

- The key to the first and second lemmata is to realize that since every reflection of a particular tiling is another tiling we are over–counting by half, modulo the self–symmetric tilings. Adding these back in and a little parity bookkeeping completes the results.
- The second lemma follows from a standard bijective double counting argument.

Self–Symmetric Fibonacci Tilings



Example: Conclusion

Theorem

The minimal order of the recurrence relation for the number of tilings of a $m \times n$ rectangle with 1×1 and 2×2 squares is at most $\mathcal{S}(f_n) - \mathcal{S}(P_n) + 1$ or

$$\frac{1}{2}(f_{2k} + f_{k+1} - P_{2k} - P_{k+2}) + 1 \tag{1}$$

when $m = 2k$, and

$$\frac{1}{2}(f_{2k+1} + f_k - P_{2k+1} - P_{k-1}) + 1 \tag{2}$$

when $m = 2k + 1$.

Example Order Bounds

Table: Comparison between the derived bound and the actual order

n	2	3	4	5	6	7	8	9	10
Order(a_n)	2	2	3	4	6	8	14	19	32
Bound	2	2	3	4	7	10	17	26	44

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What does this example show in general?

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  :)
- On the bright side...

General $1 \times n$ Case

In the preceding example, knowing two $1 \times n$ cases allowed us to reduce the upper bound from $1,156$ to 10 without a significant amount of extra effort. Here we give an expression for all $1 \times n$ rectangular tilings, where the tiles in T are allowed to have multiple colors.

Notation

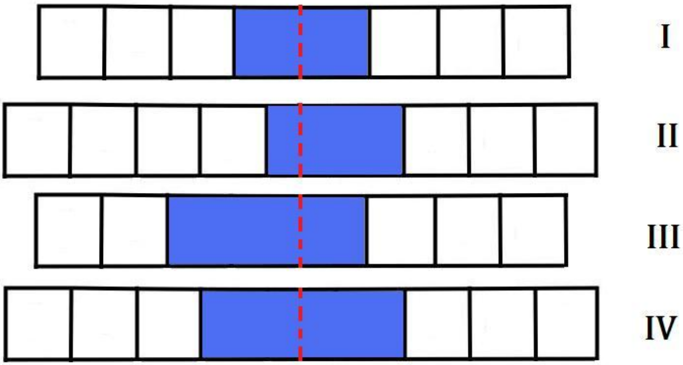
We begin by defining some convenient notation. Since we are covering boards of dimension $\{1 \times n | n \in \mathbb{N}\}$. Let $T = (a_1, a_2, a_3, \ldots)$, where a_m is the number of distinct colors of m–dominoes available. Then, T_n is the number of ways to tile a $1 \times n$ rectangle with colored dominoes in T. Connecting to our example, the Fibonacci numbers would be $T = (1, 1, 0, 0, 0, \ldots)$ while the Padovan numbers have $T = (0, 1, 1, 0, 0, 0, \ldots)$.

Coefficients

We also need to define a set of coefficients based on the parity of the domino length and the rectangle length.

$$c_j = \begin{cases} T_{n-\frac{j}{2}} & j \equiv n \equiv 0 \pmod 2 \\ 0 & j \equiv 0, n \equiv 1 \pmod 2 \\ 0 & j \equiv 1, n \equiv 0 \pmod 2 \\ T_{n-\frac{j-1}{2}} & j \equiv n \equiv 1 \pmod 2 \end{cases} \tag{3}$$

Coefficient Motivation



Complete Characterization of $1 \times n$ Tilings

Theorem

Let T be some set of colored k–dominoes, then the number of distinct tilings up to symmetry of a $1 \times n$ rectangle is equal to

$$\frac{1}{2}\left(T_n + \sum_{i=1}^{\infty} a_i c_i + \frac{T_{\frac{n}{2}}}{2} + \frac{(-1)^n T_{\frac{n}{2}}}{2}\right) \tag{4}$$

Lucas Tilings

It is natural to wonder if these methods could be adapted to give a similar formula for generalized Lucas tilings on a bracelet or necklace. Unfortunately, the complexity of the underlying symmetric groups makes this a much more complex problem. Even in the simplest case we have:

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Theorem

The number of distinct Lucas tilings of a $1 \times n$ bracelet up to symmetry is:

$$\sum_{i=0}^{\lceil \frac{n-1}{2} \rceil} \left[\frac{1}{n-i} \sum_{d|(i,n-1)} \varphi(d) \binom{\frac{n-i}{d}}{\frac{i}{d}} \right] \tag{5}$$

Larger Rectangles

As useful as this characterization is, counting distinct tilings of more general rectangles is also an interesting problem on its own. Some of the earliest motivations for this type of tiling problem arose in statistical mechanics and other applied fields where the notion of symmetric distinctness is particularly relevant. The expressions in these cases are more complex, but often give nice closed forms as well.

$k > 1$ Example

Example

The number of distinct tilings of a $3 \times n$ rectangle with squares of size 1×1 and 2×2 is

$$\frac{1}{3}\left(2^{2n-1} + 2^{n} + 2^{n-1} + \frac{1+(-1)^{n}}{2}\right) \tag{6}$$

when n is odd, and

$$\frac{1}{3}\left(2^{2n} + 2^{n} + 2^{n-1} + 1\right) \tag{7}$$

when n is even.

Symmetry Conclusion

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$$\sqrt{2^{2mn}\prod_{t=1}^{2m}\prod_{s=1}^{n}\left(\cos^2\left(\frac{s\pi}{n+1}\right)+\cos^2\left(\frac{t\pi}{2m+1}\right)\right)} \tag{8}$$

Vector Spaces

Given a particular recurrence relation $\mathcal{R}$ of order n, the set of sequences that satisfy that relation form a vector space (over $\mathbb{C}$). Since each sequence is uniquely determined by its initial conditions, the order of the space is also n.

If the roots of $\mathcal{R}$ are $\alpha_1, \alpha_2, \ldots, \alpha_k$ with respective multiplicities $m_1, m_2, \ldots, m_k$, then the set of generalized power sums (GPS) of the form:

$$\sum_{i=1}^{k} p_i(n)\alpha_i^n$$

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If we consider the set of all sequences that satisfy some LHCCRR and the set of all GPS of algebraic numbers, we see that these larger sets form a commutative ring (actually an integral domain)

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- Combinatorial interpretations (f_n) / Number–theoretic properties (F_n)

Second Order

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$$F_n = \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}$$

$$L_n = \varphi^n + \overline{\varphi}^n$$

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Identities

- Obviously, there are innumerable identities linking these two very well known sequences...
- This is exactly what we want from a basis
- What about other second order sequences?

Lucas Sequences

Lets consider the more general form of a second order recurrence relation:

$$T_n = PT_{n-1} - QT_{n-2}$$

This relation has:

- characteristic equation $x^2 - Px + Q$
- discriminant $D = P^2 - 4Q$
- roots $\alpha = \frac{P+\sqrt{D}}{2}$ and $\beta = \frac{P-\sqrt{D}}{2}$

$T_n = PT_{n-1} - QT_{n-2}$

Definition (Fundamental Lucas Sequence)

$$u_0 = 0, u_1 = 1$$

with GPS

$$u_n = \frac{\alpha^n - \beta^n}{\sqrt{D}}$$

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Definition (Fundamental Lucas Sequence)

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with GPS

$$u_n = \frac{\alpha^n - \beta^n}{\sqrt{D}}$$

Definition (Primordial Lucas Sequence)

$$v_0 = 0, v_1 = 1$$

with GPS

$$v_n = \alpha + \beta$$

Identities

Similar to the Fibonacci/Lucas identities there exist many relationships between the Lucas sequences satisfying any such recurrence relation. Particularly interesting are the following trigonometric identities:

$$u_n = \frac{2Q^{\frac{n}{2}} \sin\left(\frac{in}{2} \ln \frac{\alpha}{\beta}\right)}{\sqrt{-D}}$$

and

$$v_n = 2Q^{\frac{n}{2}} \cos\left(\frac{in}{2} \ln \frac{\alpha}{\beta}\right)$$

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Other identities

Generalized Lucas Sequences

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- Symmetric functions

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That's all...

THANK YOU