Stirling Summary

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Much of my early research in combinatorics was motivated by the problem of determining the number of cycle decompositions of a given graph family. See [2, 3] for examples and basic definitions. Recently, Amir Barghi has shown that this problem can be expressed as a type of graph factorial, where the number of cycle decompositions with a fixed number of cycles represents the kth graph Stirling number [1]. For me, the most interesting examples are those where we can compute the values for an entire sequence of graphs using a recurrence relation, as the connection of the general problem with the matrix permanent means that computing these values for arbitrary graphs is $\#P$ complete.

One natural set of examples is given by the chesspiece graphs, where we consider the adjacency graph of squares of an $m \times n$ board where two nodes are adjacent if a chess piece can legally be moved between them. Then, the following theorem tells us that recurrence relations can be constructed for a wide variety of

Theorem 1. *On any rectangular $m \times n$ board B with m fixed, and a marker on each square, where the set of permissible marker movements has a maximum horizontal displacement, the number of rearrangements of the markers on B satisfies a linear, homogeneous, constantco-efficient recurrence relation as n varies.*

This also motivates a generalization of the knight's tour problem, where we show that the knight rearrangement problem has 8,121,130,233,753,702,400 solutions.

Another natural set of examples concerns graph products. Given a specific graph product, we can ask for relations between the number of cycle decompositions of a product graph with respect to the values on the component graphs. In general, this is a difficult problem, as many of the general graph product are much denser than the input grahs. However, for spare graphs like trees, or for products like the hierarchical product or matched sum, recurrence relations can be derived. An example of this is contained in the appendix to my Ph.D. thesis [4] but I'm sure there are others.

References

[1] A. Barghi: *Stirling numbers of the first kind for graphs*, Australas. J. Combin. 70, 253268, (2018).

[2] D. DeFord: *Counting Rearrangements on Generalized Wheel Graphs*, Fibonacci Quarterly, 51(3), 259-273, (2013).

[3] D. DeFord: *Seating Rearrangements on Arbitrary Graphs*, Involve, 7(6), 787-805, (2014).

[4] D. DeFord: *Matched Products and Dynamical Models for Multiplex Networks*, Ph.D. Dissertation, Dartmouth College, (2018).