Èdouard Lucas:

The theory of recurrent sequences is an inexhaustible mine which contains all the properties of numbers; by calculating the successive terms of such sequences, decomposing them into their prime factors and seeking out by experimentation the laws of appearance and reproduction of the prime numbers, one can advance in a systematic manner the study of the properties of numbers and their application to all branches of mathematics.

Preview

Hardness results for sampling connected graph partitions with applications to redistricting

Daryl DeFord

MIT – CSAIL
Geometric Data Processing Group

Applied Math Seminar
University of Massachusetts Lowell
September 17, 2019

Outline

Collaborators

- Prof. Justin Solomon **MIT CSAIL**
- Lorenzo Najt **Wisconsin Math**
- Prof. Moon Duchin **Tufts Math**

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- Prof. Justin Solomon **MIT CSAIL**
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- *Complexity and Geometry of Sampling Connected Graph Partitions* (with L. Najt and J. Solomon), arXiv: 1908.08881.
- *ReCombination: A family of Markov chains for redistricting* (with M. Duchin and J. Solomon), preprint.
- *Competitiveness Measures for Evaluating Districting Plans* (with M. Duchin and J. Solomon), preprint.

MORAL #1:

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Computational Redistricting is NOT a solved problem!

☹

MORAL #2:

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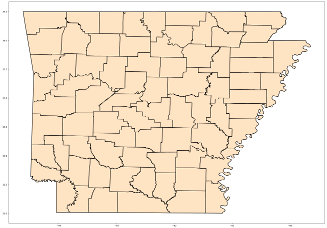
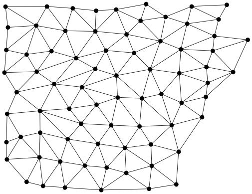
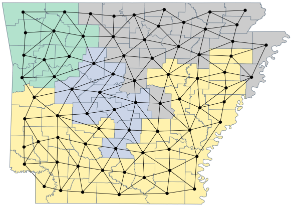
☺

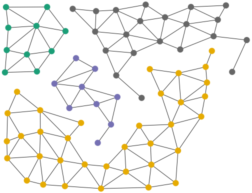
Political Partitioning



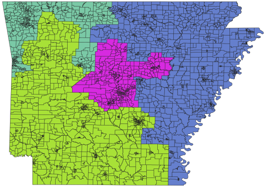


Political Partitioning

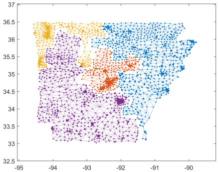




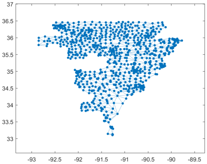
Arkansas Congressional Districts



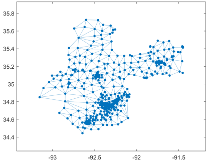
Geography



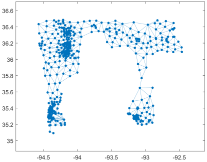
Dual Graph



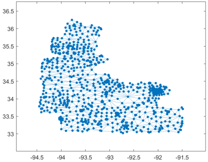
District # 1



District # 2



District # 3



District # 4

Permissible Districting Plans

We want to partition a given geography (graph), at a given scale, into k pieces, satisfying some constraints:

- Contiguity
- Population Balance
- Compactness
- Communities of Interest
- Municipal Boundaries
- Competitiveness/Responsiveness
- Incumbency Protection
- ...

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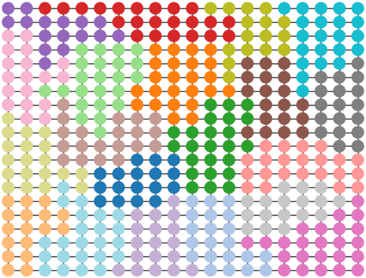
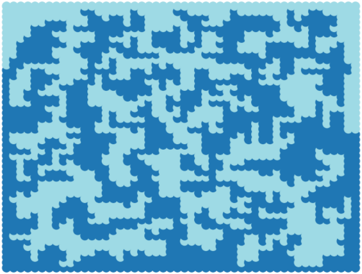
Mathematical Formulation

Given a (connected, planar) graph $G = (V, E)$:

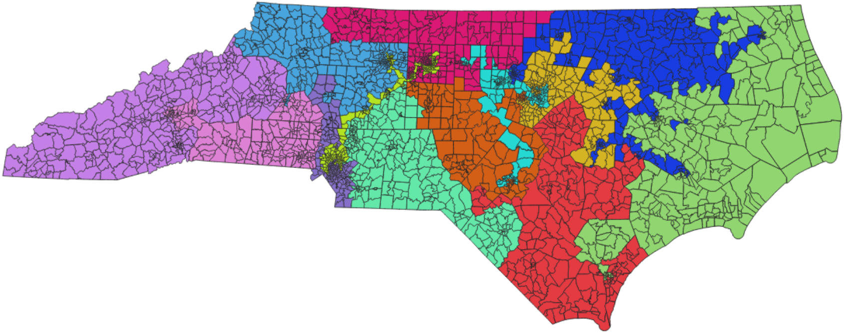
- A k-**partition** $P = \{V_1, V_2, \dots, V_k\}$ of G is a collection of disjoint subsets $V_i \subseteq V$ whose union is V.
- A partition P is **connected** if the subgraph induced by V_i is connected for all i.
- The **cut edges** of P are the edges (u, w) for which $u \in V_i$, $w \in V_j$, and $i \neq j$
- A partition P is ε-**balanced** if $\mu(1 - \varepsilon) \leq |V_i| \leq \mu(1 + \varepsilon)$ for all i where μ is the mean of the $|V_i|$'s
- An **equi–partition** is a 0-balanced partition

(Discrete) Total Perimeter





Ugly Shapes



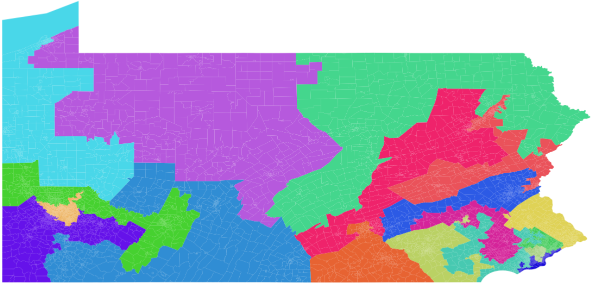
Ugly Shapes

NC12 #1

NC12 #2

NC12 #12

Ugly Shapes



Measurement Problems

Theorem (Bar-Natan, Najt, and Schutzman 2019[1])

There is no local homeomorphism from the globe to the plane that preserves your favorite compactness measure.

Problem (Barnes and Solomon 2018[2])

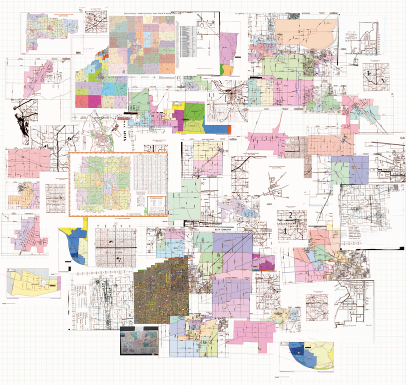
geographic Compactness scores can be distorted by:

- *Data resolution*
- *Map projection*
- *State borders and coastline*
- *Topography*
- ...

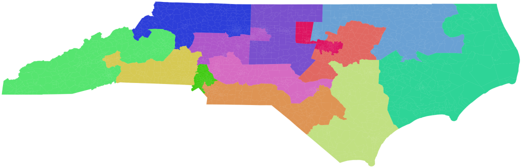
[1] The Gerrymandering Jumble: Map Projections Permute Districts' Compactness Scores, arXiv:1905.03173

[2] Gerrymandering and Compactness: Implementation Flexibility and Abuse, arXiv:1803.02857

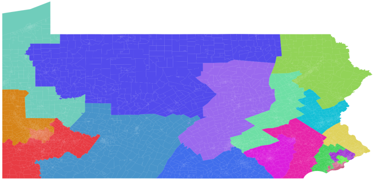
Data Availability



Partisan Imbalance



NC16



PA TS-Proposed

Electoral Data

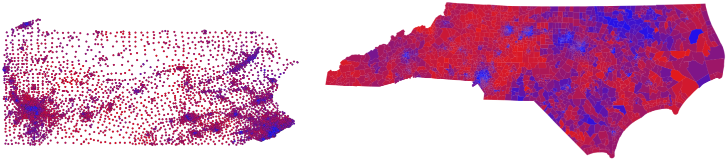
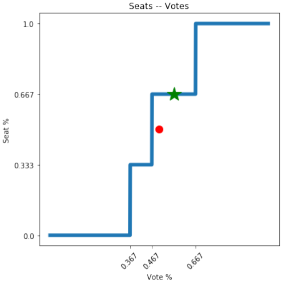
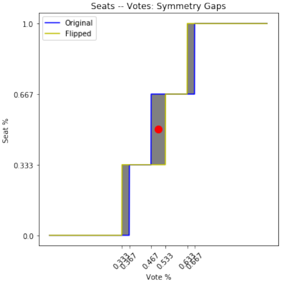


Figure: 2016 Presidential election votes by precinct in PA and NC.

Partisan Metrics

- Number of seats (proportionality)
- Mean-Median score
- Partisan Gini
- Efficiency gap





Partisan Fairness

- MA
  - Duchin et al. (2018) Locating the representational baseline: Republicans in Massachusetts arXiv:1810.09051
  - Not all partisan outcomes are possible, given discretization
- MD
  - Two 2018 preprints claiming not gerrymandered
  - Court then ruled one district unconstitutional
- NC/PA/WI
  - Heavy court involvement
  - Wide variance in partisan metrics

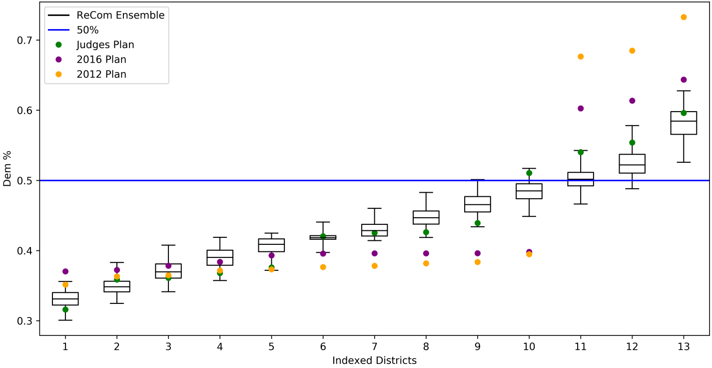
MORAL:

Computational Redistricting is **NOT** a solved problem!

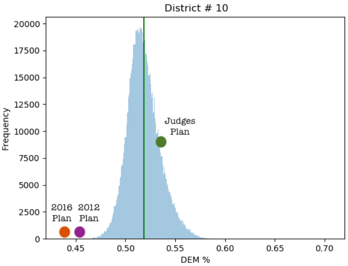
Ensemble Analysis

- The wide variety in rules applied to districting problems (even in the same state) means that any single measure of gerrymandering will be insufficient/exploitable
- Instead we want to compare to large ensembles of other feasible plans.
- This allows us to understand the impacts of the underlying political and demographic geography on a wide collection of metrics.

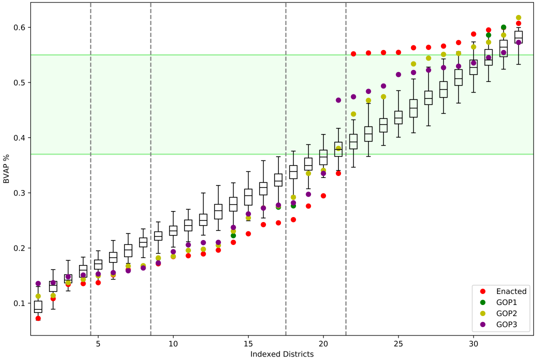
Outlier Example: NC



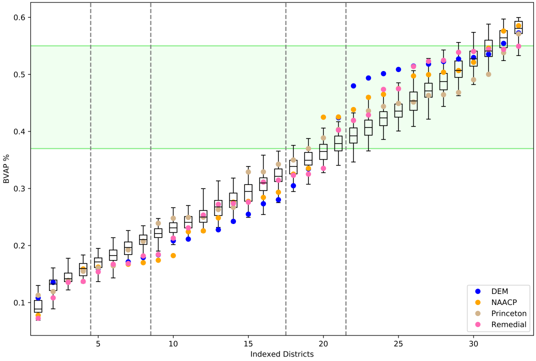
Outlier Example: NC



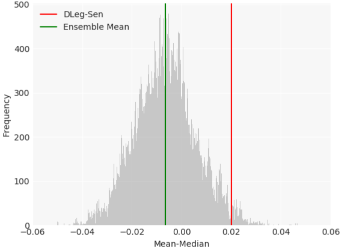
Outlier Example: VA



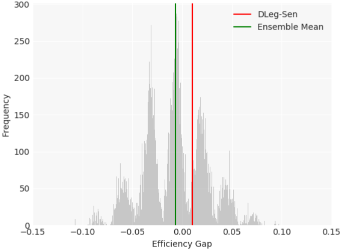
Outlier Example: VA



Baseline Example: VA

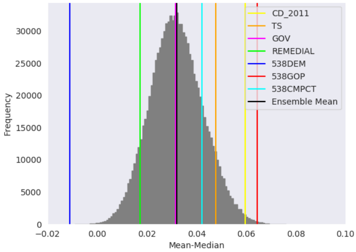


Mean–Median

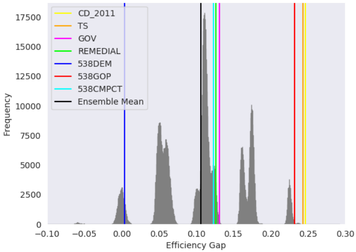


Efficiency Gap

Baseline Example: PA

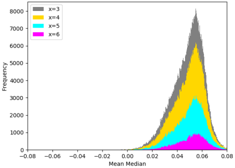


Mean–Median

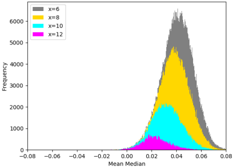


Efficiency Gap

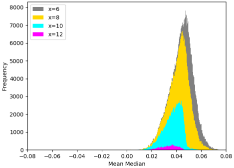
Reform Example: Competitiveness



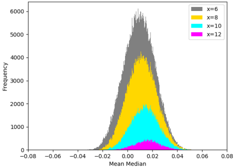
UT



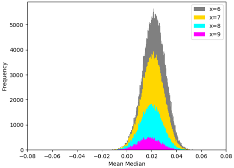
GA



WI



VA



MA

Which ensembles?

Ensembles in Practice

- The appeal of an ensemble method is that you get to control the input data very carefully
- However, just because a particular type of data was not considered doesn't mean that the outcome is necessarily "fair"
- There are lots of "random" methods for constructing districting plans
- Most don't offer any control over the distribution that you are drawing from

MCMC on partitions

1. Set constraints to define the state space
2. Start with an initial plan
3. Propose a modification
4. Verify that the modification satisfies the constraints
5. Accept using MH criterion
6. Repeat

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Why?

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Why?

- Control over sampling distribution and input data
- Possibility of local sampling
- Ergodic Theorem

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Why?

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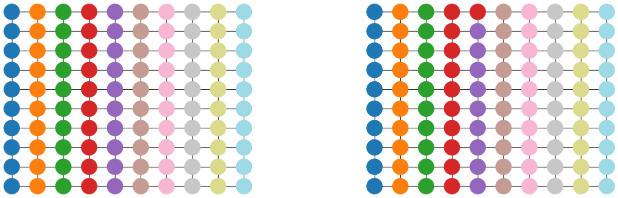
Single Edge Flip Proposals

1. Uniformly choose a cut edge
2. Change one of the incident node assignments to the other

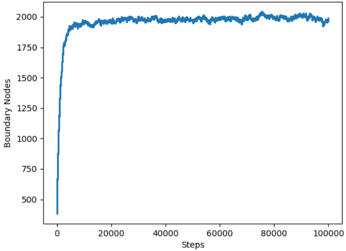


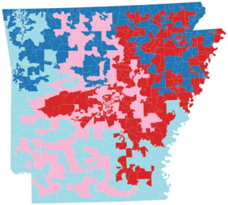
- Mattingly et al. (2017, 2018) Court cases in NC and WI.
- Pegden et al. Assessing significance in a Markov chain without mixing, PNAS, (2017). Court case in PA.

Single Edge Ensembles

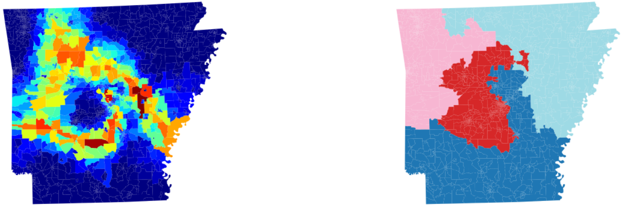
PA Single Edge Flip

Unconstrained Flip

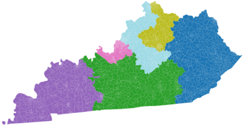


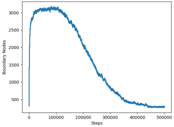


Constrained Flip



Annealing

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Boundary Nodes
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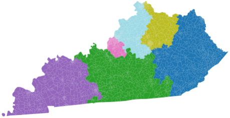

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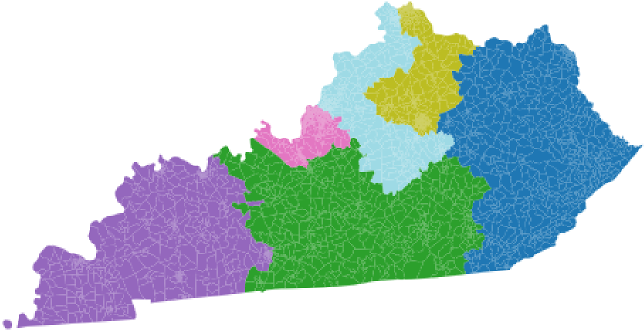

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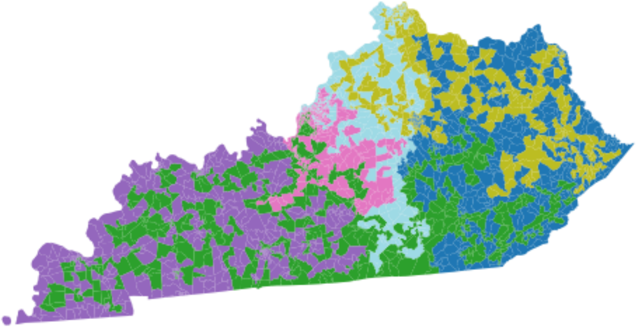

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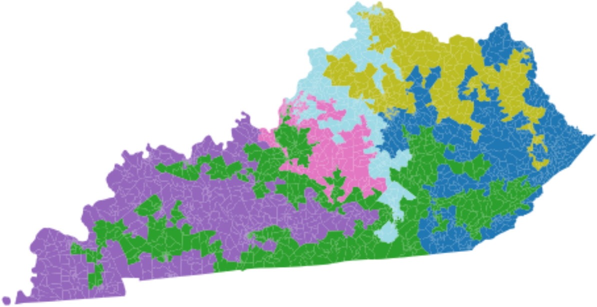

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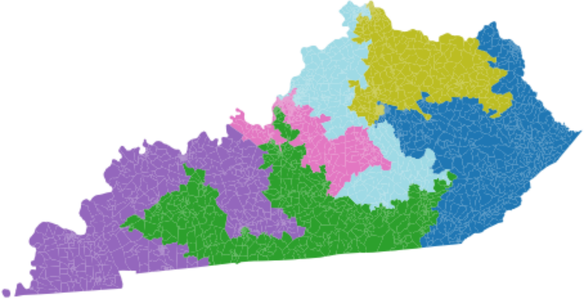

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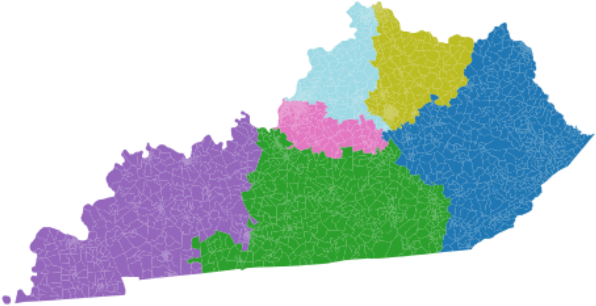

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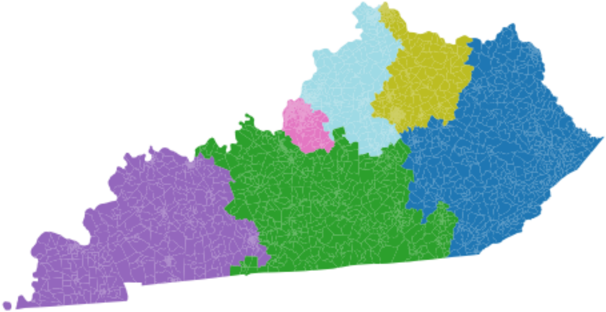

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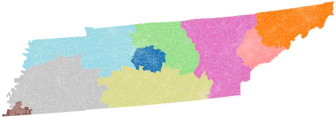

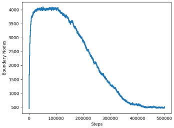
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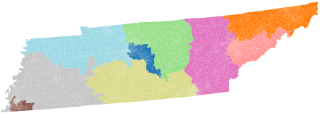

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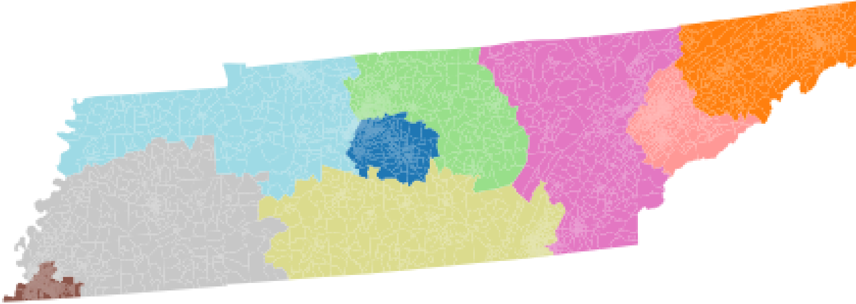

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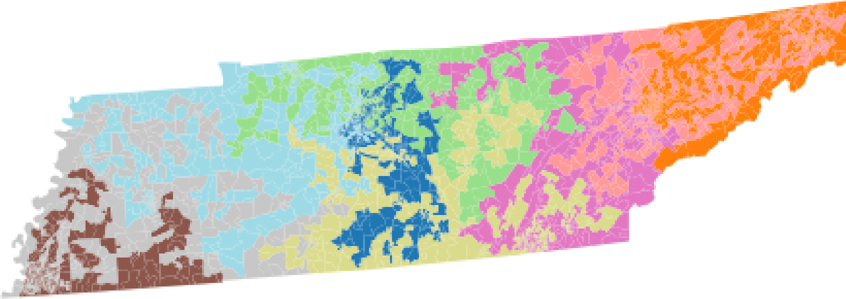

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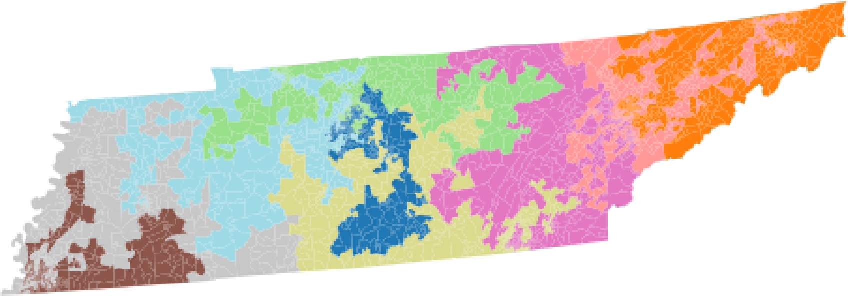

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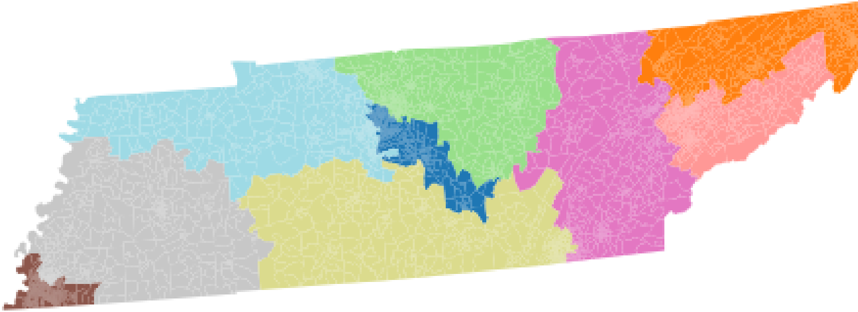

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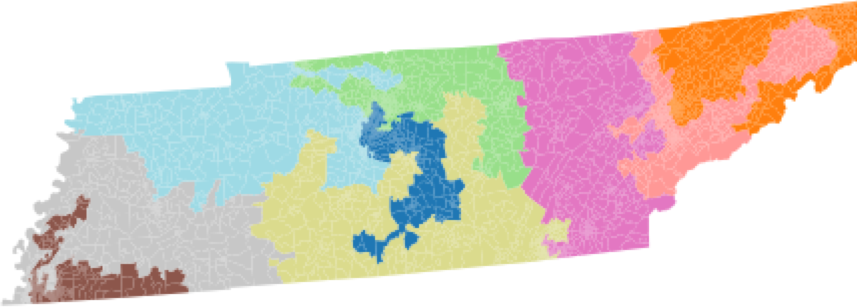

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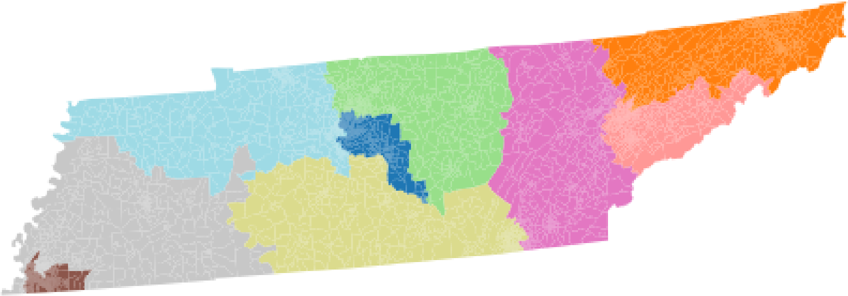

Uniform Sampling of Contiguous Partitions

Theorem (Najt, D., and Solomon 2019)

Suppose that $\mathscr{C}$ is the class of connected planar graphs and $k \geq 2$. If there is a polynomial time algorithm to sample uniformly from:

- *the connected k-partitions of graphs in $\mathscr{C}$,*
- *or the connected, 0-balanced k-partitions of graphs in $\mathscr{C}$.*

then $RP = NP$.

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Remark

This theorem has various interesting extensions, including:

- *Connectivity constraints on $\mathscr{C}$*
- *Degree bounds*
- *Distributions proportional to cut length*
- *TV distribution approximation*

Stronger Version Example

Theorem (D., Najt, and Solomon 2019)

Let $\mathscr{C}$ be the class of cubic, planar 3-connected graphs, with face degree bounded by $C = 60$. Let $\mu_x(G)$ be the probability measure on $P_k(G)$ such that a partition P is drawn with probability proportional to $x^{cut(P)}$. Fix some $x > 1/\sqrt{2}$, $\epsilon > 0$ and $\alpha < 1$. Suppose that there was an algorithm to sample from $P_2^\epsilon(G)$ according to a distribution $\nu(G)$, such that $||\nu_G - \mu_x(G)||_{TV} < \alpha$, which runs polynomial time on all $G \in \mathscr{C}$. Then $RP = NP$.

Proof Outline Sketch

Following technique of Jerrum, Valiant, and Vazirani[1].

1. Show that uniformly sampling simple cycles is hard on some class $\mathscr{C}$
   1. Choose a gadget that respects $\mathscr{C}$ and allows us to concentrate probability on long cycles
   2. Count the proportion of cycles as a function of length
   3. Reduce to Hamiltonian path on the graph class
2. Show closure of class under planar dual
3. Identify partitions with cut edges $\mapsto$ simple cycles (via planar duality)
4. Conclude that sampling partitions would allow you to sample from cycles which would allow you to find Hamiltonian cycles

□

[1] M. Jerrum, L. Valiant, and V. Vazirani, Random generation of combinatorial structures from a uniform distribution, Theoretical Computer Science, 43 (1986), 169–188.

Proof Sketch – Planar 2–Partitions

Still following technique of Jerrum, Valiant, and Vazirani.

1. Let $\mathscr{C}$ be the planar connected graphs
   1. Replace the edges with chains of dipoles
   2. Hamiltonian hardness for $\mathscr{C}$ given by [1]
2. $\mathscr{C}$ closed under planar duals
3. Identify partitions with cut edges (via planar duality)

□

[1] M. Garey, D. Johnson, and R. Tarjan, The Planar Hamiltonian Circuit Problem is NP-Complete, SIAM Journal on Computing, 5, (1976), 704–714.

Slowly Mixing Graph Families

Theorem (Najt, D., and Solomon 2019)

Let G be any connected graph. Then let $G^{(d)}$ be the graph obtained by replacing each edge by a doubled d-star. Then the flip walk on partitions of family of graphs $G^{(d)}_{d \geq 1}$ is slowly mixing, in the sense the Cheeger constant is decaying exponentially fast. More specifically:

$$H(Partition\ Graph(G^{(d)}) = O(2^{-d})$$

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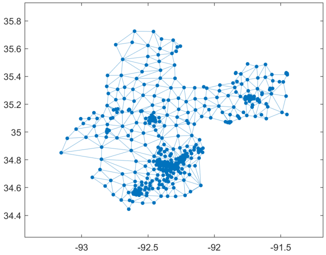
Remark

There are many similar constructions that give rise to equivalent mixing results.

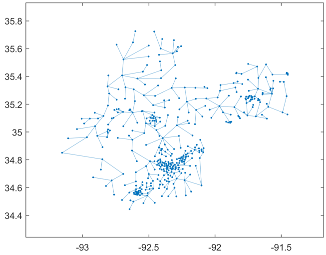
Slow Mixing Example

Slow Mixing Example

Tree based methods



District



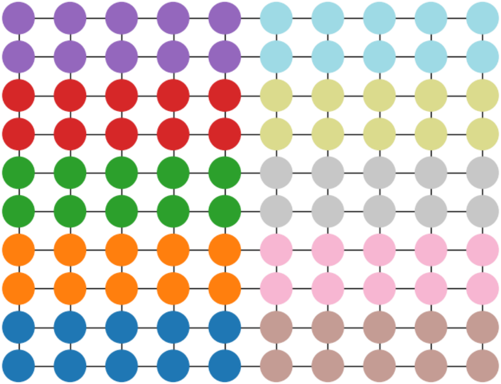
Spanning Tree

Tree Seeds Ensemble

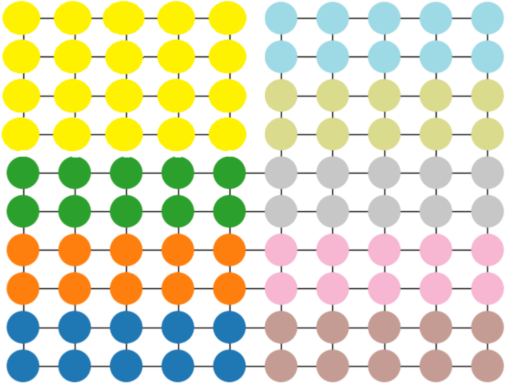
Recombination Steps

1. At each step, select two adjacent **districts**
2. Merge the subunits of those two districts
3. Draw a spanning tree for the new super–district
4. Delete an edge leaving two population balanced districts
5. Repeat
6. (Optional) Mix with single edge flips

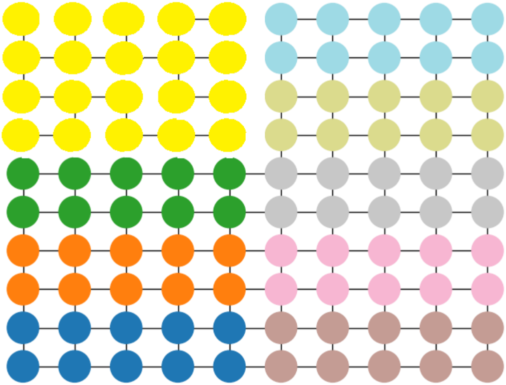
Recombination Step Example



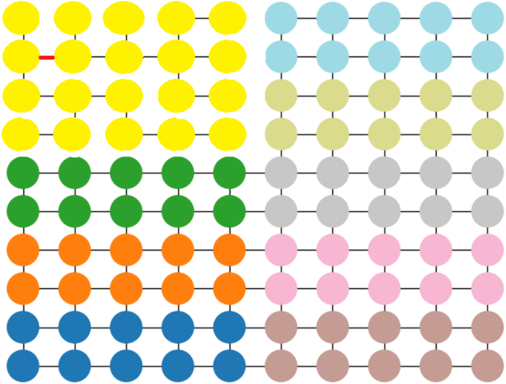
Recombination Step Example



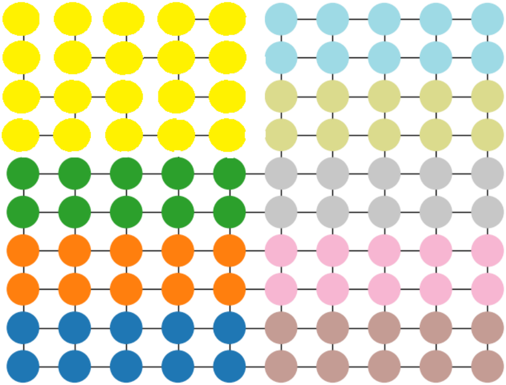
Recombination Step Example



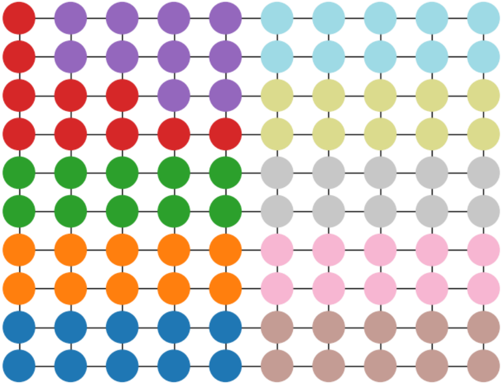
Recombination Step Example



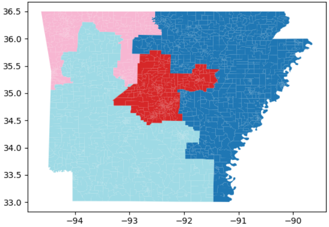
Recombination Step Example



Recombination Step Example



AR Ensembles



PA Recombination Steps

General Tree Proposals

1. Form the induced subgraph on the complement of the cut edges
2. Add some subset of the cut edges
3. Uniformly select a maximal spanning forest
4. Apply a Markov chain on trees
5. Partition the spanning forest into k population balanced pieces

Special Cases

- Uniform Trees: Add all cut edges
- k–edges: Uniformly add k cut edges
- Recombination: Add all cut edges between one pair of districts.
- Super-Recombination: Take a maximal matching on the dual graph to the districts and add all cut edges between matched districts.
- Bounce Walk: Add a single cut edge between enough pairs of districts to make a tree in the dual graph of districts.

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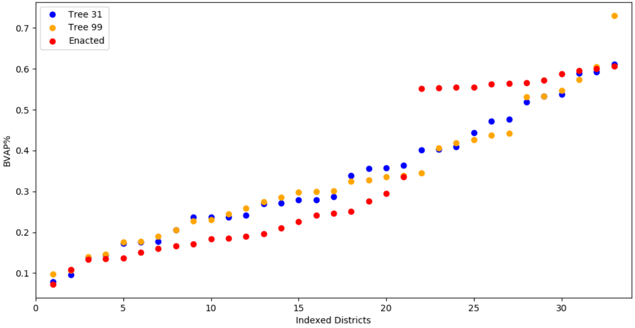
Question

What are the steady state distributions (and mixing times) of these walks?

Tree Partitioning Questions

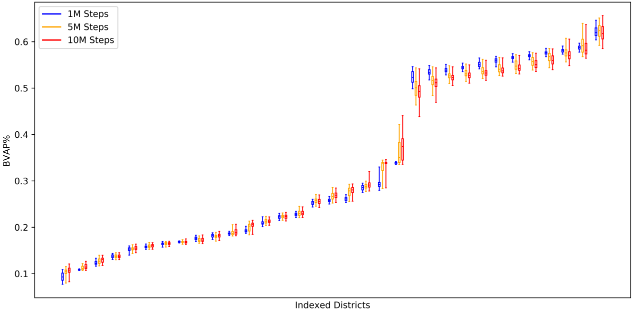
- Characterizing the distribution on partitions defined by cutting trees!
- How bad is the best cut?
- Criteria for determining when a tree is ε cuttable?
- Criteria for determining when all spanning trees of a graph are ε cuttable?
- How hard is it to find the mininum ε for which a cut exists?
- As a function of ε what proportion of spanning trees are cuttable?
- As a function of ε what proportion of edges in a given tree are cuttable?
- What is the fastest way to sample uniformly from $k-1$ balanced cut edges?

Initial Seeds

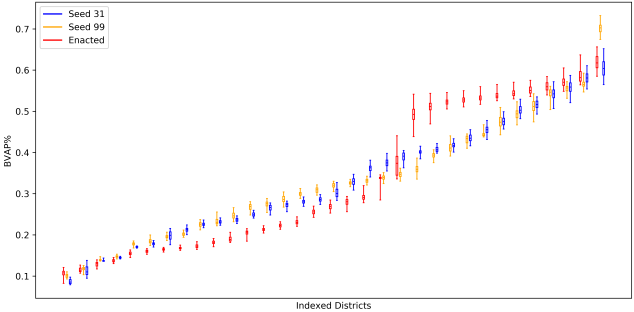


Initial

Boundary Flip Mixing – Length

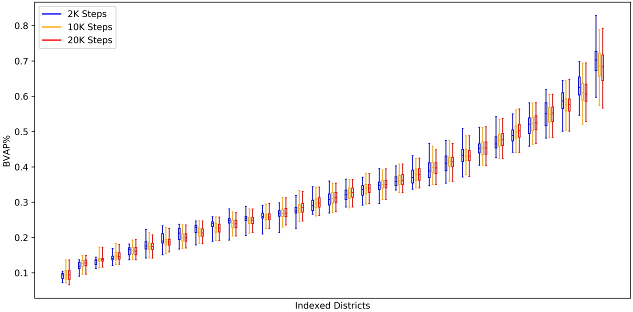


Boundary Flip Mixing – Seeds



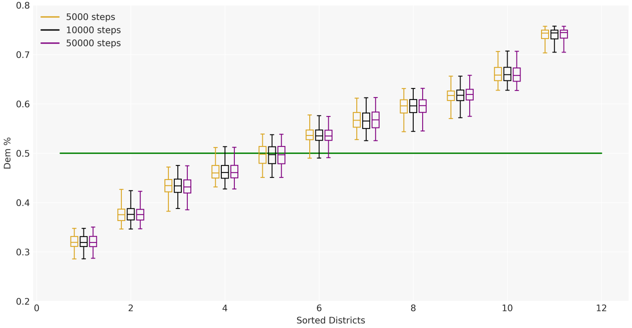
10,000,000 Flip Steps

Recombination Mixing – Length



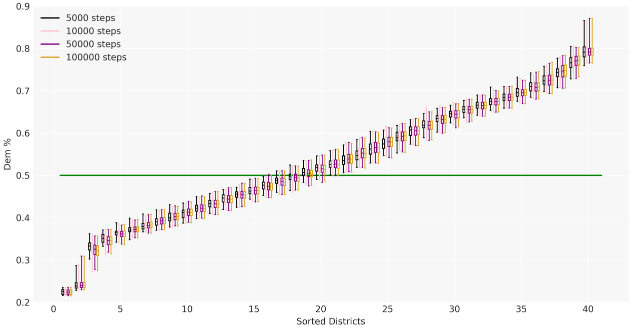
20,000 Recombination Steps

Recombination Mixing – Length



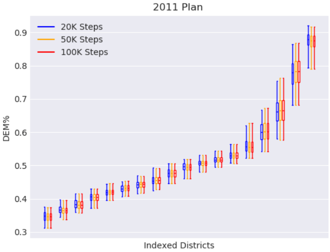
50,000 Recombination Steps

Recombination Mixing – Length

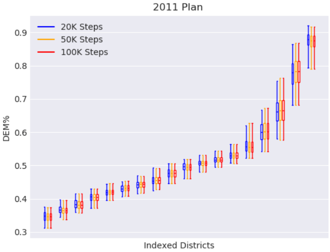


50,000 Recombination Steps

Recombination Mixing – Length

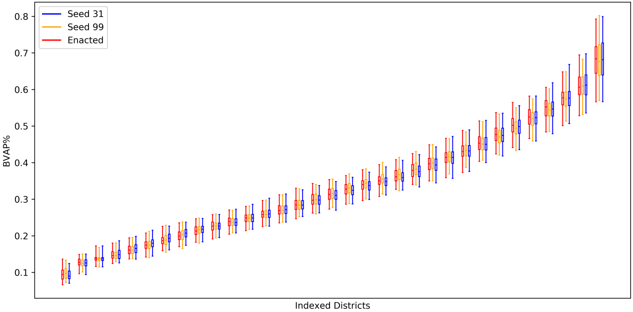


2011 Seed



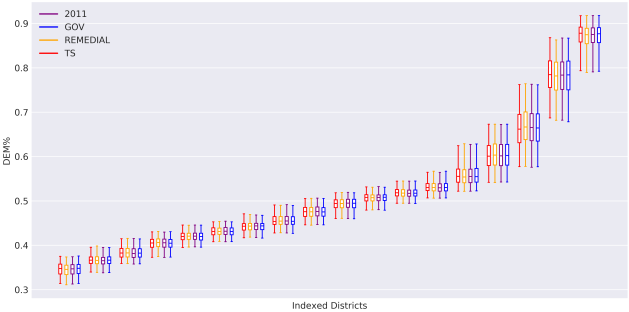
GOV Seed

Recombination Mixing – Seeds



20,000 Recombination Steps

Recombination Mixing – Seeds



20,000 Recombination Steps

MORAL:

Computational Redistricting is NOT a solved problem!

Try it at home!

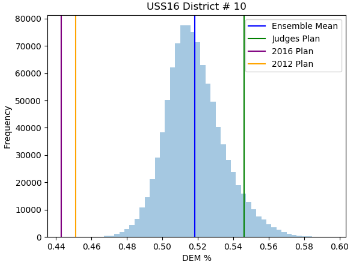
- Draw your own districts with **Districtr**
  - `https://districtr.org`
  - Easy to generate complete districting plans in browser or on a tablet
  - Measures district demographics and expected partisan performance
  - Identifies communities of interest
- Generate your own ensembles with **GerryChain**[1]
  - `https://github.com/mggg/gerrychain`
  - Flexible, modular software for sampling graph partitions
  - Handles the geodata processing as well as the MCMC sampling
  - Current support for a
  - Successfully applied in VA, NC, PA, etc.

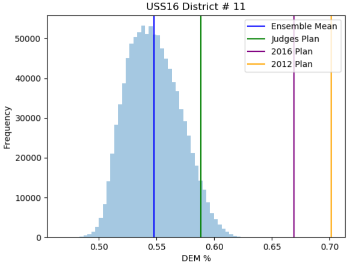
[1]Originally RunDMCMC

The End

Thanks!

Ensemble Example: NC





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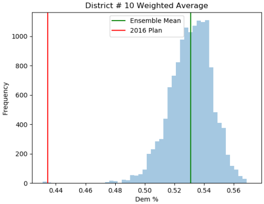
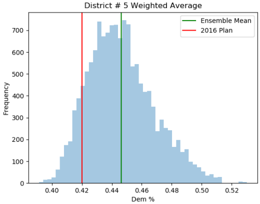
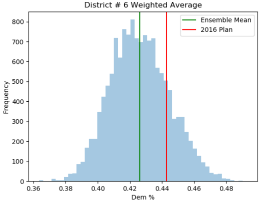
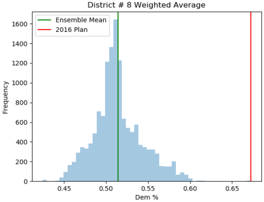
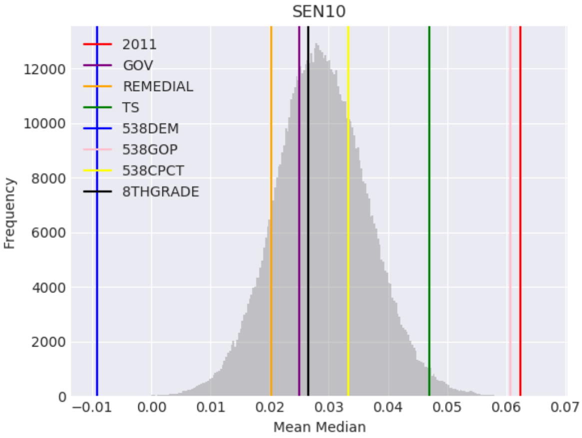








Ensemble Example: PA



Ensemble Example: PA