Èdouard Lucas:

The theory of recurrent sequences is an inexhaustible mine which contains all the properties of numbers; by calculating the successive terms of such sequences, decomposing them into their prime factors and seeking out by experimentation the laws of appearance and reproduction of the prime numbers, one can advance in a systematic manner the study of the properties of numbers and their application to all branches of mathematics.

Enumerating Distinct Chessboard Tilings

Daryl DeFord

Dartmouth College
Department of Mathematics

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and Their Applications
Rochester Institute of Technology

Abstract

Counting the number of distinct colorings of various discrete objects, via Burnside's Lemma and Pòlya Counting, is a traditional problem in combinatorics. We address a related question for more general tiling situations: Given an $m \times n$ chessboard and a fixed set of (possibly colored) tiles, how many distinct tilings exist, up to symmetry? More specifically, we are interested in the recurrent sequences formed by counting the number of distinct tilings of boards of size $(m \times 1), (m \times 2), (m \times 3), \ldots$, for a fixed set of tiles and some natural number m. The terms of these sequences can be used to construct upper bounds on the orders of recurrences satisfied by other classes of tiling problems not reduced by symmetry.

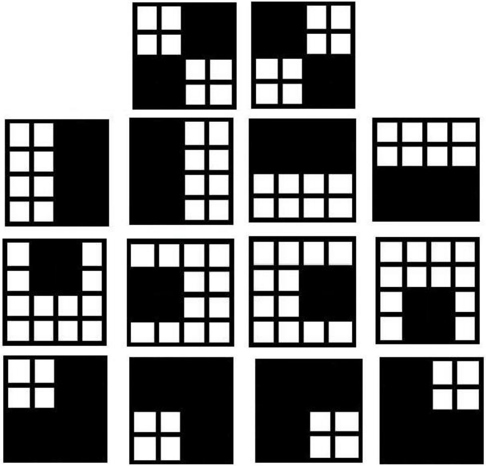
We present explicit results and closed forms for several well–known classes of tiling problems, including domino tilings and tilings with squares of arbitrary sizes. Several of these cases have convenient representations in terms of the combinatorial Fibonacci numbers. Finally, we give a characterization of all $1 \times n$ tiling problems in terms of the generalized Fibonacci numbers and colored Fibonacci tilings.

Dartmouth

Outline

1. Introduction
2. Chessboard Tilings
3. Recurrence Orders
4. Chessboard Rearrangements
5. Motivating Example
6. Results
7. References
8. Acknowledgments

Chessboard Tilings



Motivation

- Fibonacci and Lucas Tilings
- Why?

$$f_m f_n + f_{m-1} f_{n-1} = f_{m+n} \quad (1)$$

$$\sum_j \binom{n-j}{j} = f_n \quad (2)$$

$$\sum_i \sum_j \binom{n-i}{j} \binom{n-j}{i} = \sum_{k=0}^{n} f_{2k} \quad (3)$$

- $1 \times n$ Tilings [1]

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$$
\begin{aligned}
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\sum_j \binom{n-j}{j} &= f_n && (2)\\
\sum_i \sum_j \binom{n-i}{j}\binom{n-j}{i} &= \textstyle\sum_{k=0}^{n} f_{2_k} && (3)
\end{aligned}
$$

- $1 \times n$ Tilings [1]

Tiling Recurrences

Definition

Let T be a fixed set of tiles and $m \geq 1$ an integer. Construct a sequence $\{t_n\}$ by defining the n^{th} term to be the number of ways to tile a $m \times n$ board with tiles in T.

Theorem (Webb, Criddle, DeTemple [9])

For all sets T the sequence defined above satisfies a linear, homogeneous, constant coefficient recurrence relation.

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Methodology

(Proof Sketch).

Consider all of the possible ways to cover the initial column with tiles in T and construct a linear system (in the successor operator) of relations between the resulting sequences. The determinant of this system is the characteristic polynomial of an annihilating recurrence relation (not necessarily minimal). $\square$

Sequences of Sequences

- This existence proof leads to a “natural” and “accurate” upper bound of 2^{md} on the order of the recurrence relation satisfied by $\{t_n\}$ for arbitrary T.
- In particular, I am interested in the sequence of sequences that is formed from a fixed tile set T by letting the number of rows in the board vary.
- That is, consider the related family of sequences $\{t_n^m\}$ and their respective recurrence order bounds.

Why Recurrence Orders?

- Identities [3]
  - How many cases to check?
- Initial Conditions
  - Computational Feasibility

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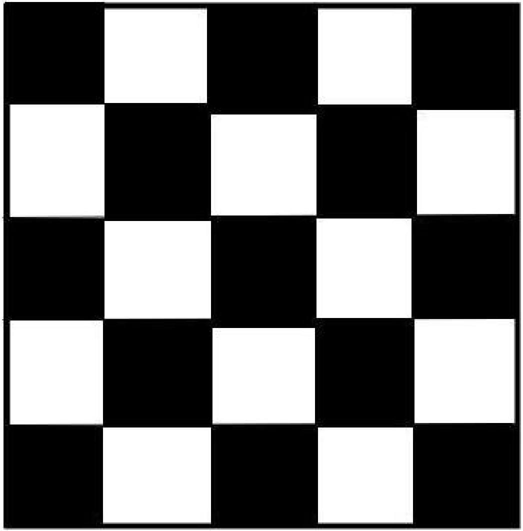
"Classroom"



Original Problem (Honsberger)[5]

Problem

A classroom has 5 rows of 5 desks per row. The teacher requests each pupil to change his seat by going either to the seat in front, the one behind, the one to his left, or the one to his right (of course not all these options are possible to all students). Is it possible to carry out his directive?

Solution ([6, 8])

No. However, Cooper et al. gave several interesting counting generalizations with Fibonacci relations.

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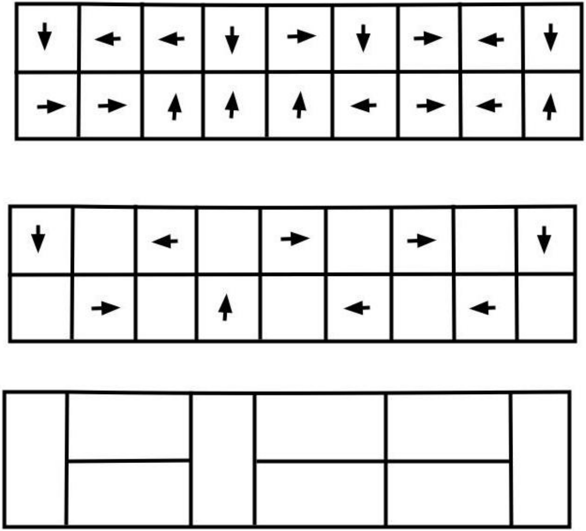
Seating Rearrangements and Tilings



Rules

In order to count more general rearrangements on chessboards, we constructed the following problem statement:

Definition

Given a $m \times n$ board with a single marker on each square decide on a set of permissible moves. We want to count the number of legitimate "rearrangements" of these markers subject to the following rules:

- Each marker must make one "move".
- After all of the markers have moved, each square must contain exactly one marker.

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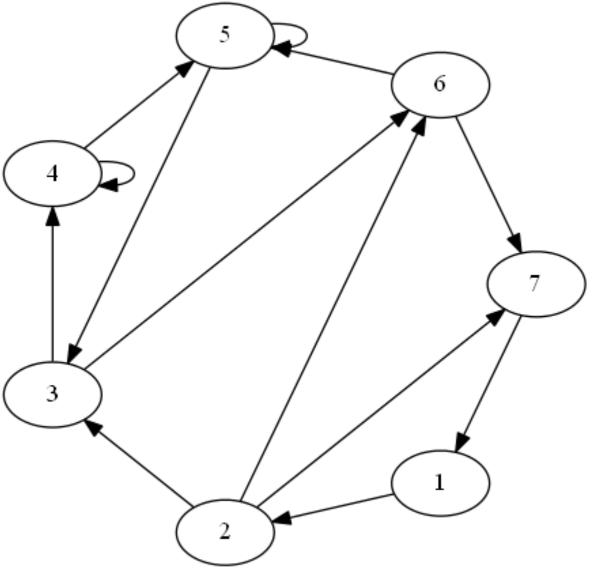
Arbitrary Graphs



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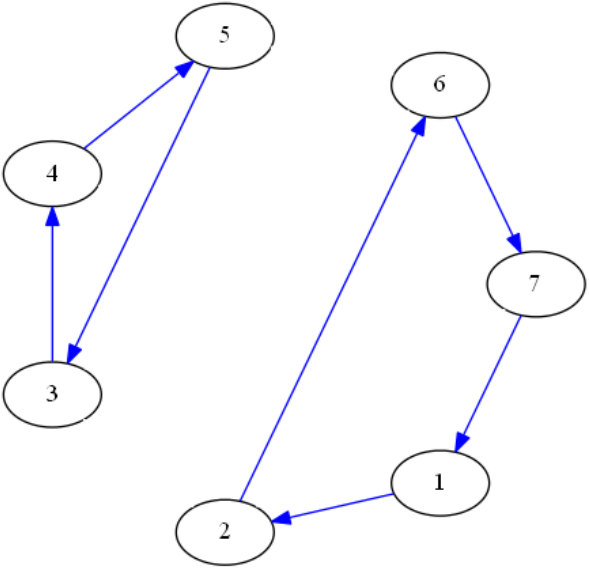
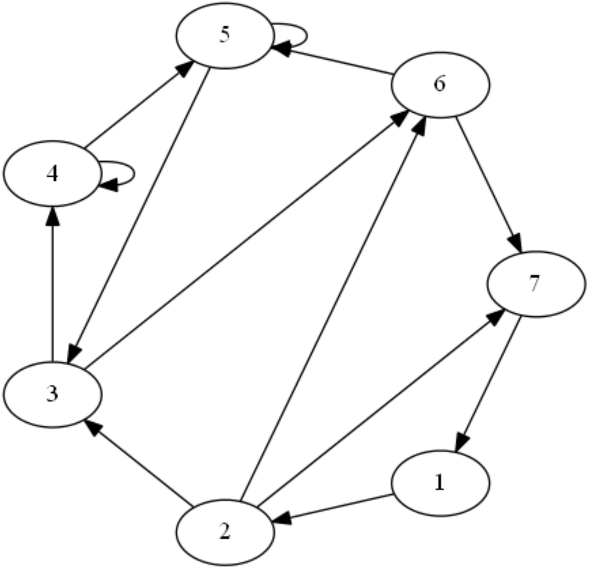


Arbitrary Graphs



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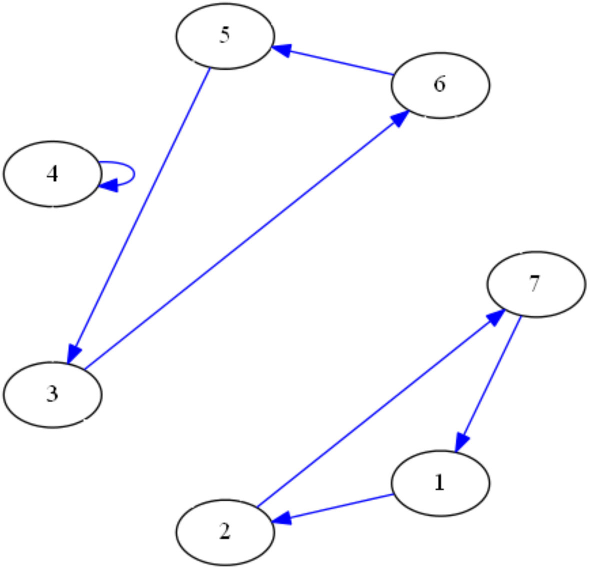


Simple Graphs

Graph	Rearrangements	With Stays
P_n	$0, 1, 0, 1, 0...$	f_n
C_n	$0, 1, 2, 4, 2, 4...$	$l_n + 2 = f_n + f_{n-2} + 2$
K_n	$D(n)$	$n!$
$K_{n,n}$	$(n!)^2$	$\sum_{i=0}^{n} \left[(n)_i\right]^2$
$K_{m,n}$ with $m \leq n$	0	$\sum_{i=0}^{m} (m)_i (n)_i$

More Complex Graphs [2]

- Wheel Graphs:

$$nf_{n+2} + f_n + f_{n-2} - 2n + 2$$

- Flat Wheel Graphs:

$$f_n + \sum_{l=1}^{n} \left[\left(f_{n-l} \sum_{j=0}^{l-2} [f_j] \right) + (f_{l-1} f_{n-l}) + \left(f_{l-1} \sum_{k=0}^{n-l-1} [f_k] \right) \right]$$

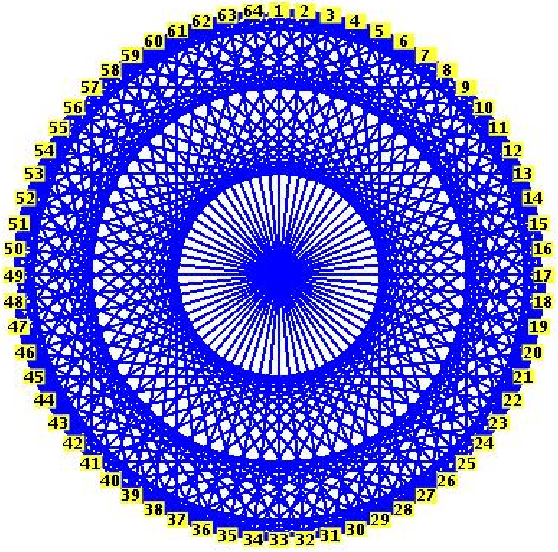
- Flower Graphs:

$$l_{(k-2)n} + 2 + nf_{(k-2)n-1} + 2nf_{(n-2)k-(n-1)} + 2n \sum_{i=1}^{k-2} f_{(k-2)(n-i-1)-1}$$

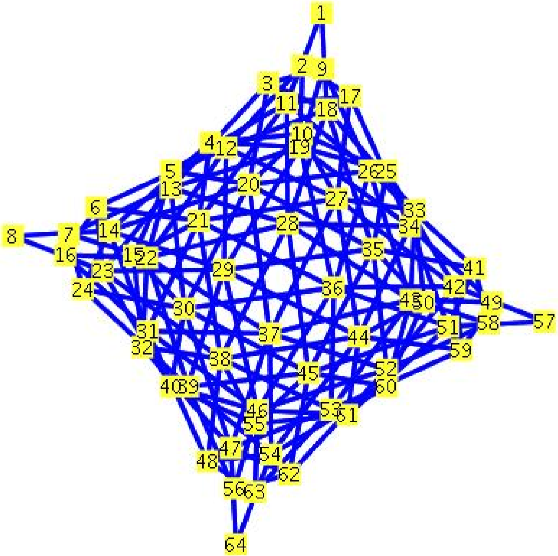
- Dutch Windmills:

$$(f_{n-1})^m + 2m(f_{n-2} + 1)(f_{n-1})^{m-1}$$

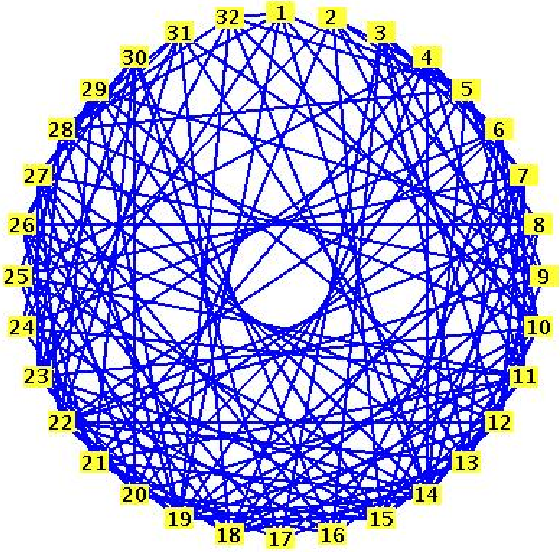
8×8 Rook Graph



8×8 Knight Graph



8×8 Bishop Graph



Chesspiece Rearrangements

- $1 \times n$
  - Kings: f_n
  - Queens/Rooks: $n!$
  - Bishops/Knights: 1
- $2 \times n$
  - Kings: $a_n = 6a_{n-1} + 12a_{n-2} - 16a_{n-2}$
  - Bishops: f_n^2
  - Knights: f_n^4 and $f_n^2 f_{n-1}^2$
  - Rooks: $\sum_{i=0}^{n} \binom{n}{i}((n-i)!)^2$

Rearrangement Recurrences

We may proceed as in the case of tilings to define sequences corresponding to a fixed set of permissible moves and fixed number of rows.

Theorem (D.)

Let $m \geq 1$ be an integer. For any fixed set of permissible moves such that there is an upper bound on the horizontal displacement of each piece, the sequence $\{r_n\}$ defined by the number of legitimate rearrangements on a $m \times n$ board satisfies a linear, homogeneous, constant coefficient recurrence relation.

Rearrangement Recurrence Orders

The bounds derived from the proof of this theorem are 4^{md}, even worse than the tiling case. Additionally, computational feasibility is an even larger issue.

Kings						
m	1	2	3	4	5	6
Bound	4	16	64	256	1024	4096
Order	2	3	10	27	53	100+

Knights				
m	1	2	3	4
Bound	4	16	256	4096
Order	1	8	27	lots

Motivating Example

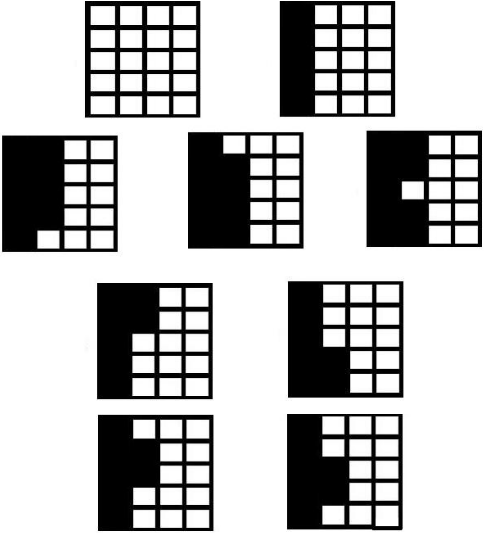
Example

Let $T = \{[1 \times 1], [2 \times 2]\}$.

Table : Toy Example

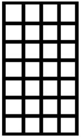
m	1	2	3	4	5	6	7	8	9	10
Upper Bound	1	4	9	25	64	169	441	1,156	3,025	7,921
$\mathcal{O}(T_n)$	1	2	2	3	4	6	8	14	19	32

Preliminary Observations

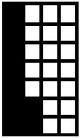


$m = 7$

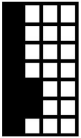
a_n



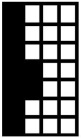
b_n



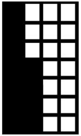
c_n



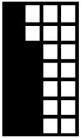
d_n



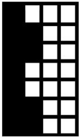
e_n



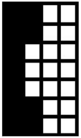
f_n



g_n



h_n



Simple Symmetries

Lemma

The number of endings with no consecutive 1×1 tiles is equal to P_{n+2}.

Lemma

The number of distinct Fibonacci tilings $\mathcal{S}(f_n)$ of order n up to symmetry is equal to $\frac{1}{2}(f_{2k} + f_{k+1})$ when $n = 2k$ and $\frac{1}{2}(f_{2k+1} + f_k)$ when $n = 2k + 1$.

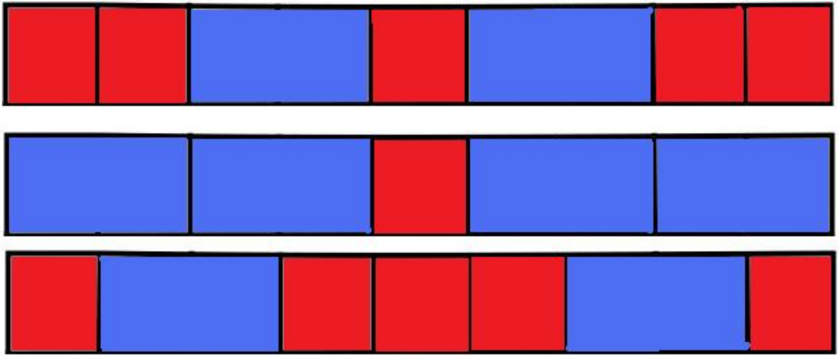
Lemma

The number of distinct Padovan tilings $\mathcal{S}(P_n)$ of order n up to symmetry is equal to $\frac{1}{2}(P_{2k} + P_{k+2})$ when $n = 2k$ and $\frac{1}{2}(P_{2k+1} + P_{k-1})$ when $n = 2k + 1$.

Lemma Proofs

- The first lemma follows from a standard bijective double counting argument.
- The key to the remaining lemmas is to realize that since every reflection of a particular tiling is another tiling we are over–counting by half, modulo the self–symmetric tilings. Adding these back in and a little parity bookkeeping completes the results.

Self–Symmetric Fibonacci Tilings



Example: Conclusion

Theorem

The minimal order of the recurrence relation for the number of tilings of a $k \times n$ rectangle with 1×1 and 2×2 squares is at most $\mathcal{S}(f_n) - \mathcal{S}(P_n) + 1$.

Table : Toy Example

m	1	2	3	4	5	6	7	8	9
Upper Bound	1	4	9	25	64	169	441	1,156	3,025
$\mathcal{S}(f_n) - \mathcal{S}(P_n) + 1$	1	2	2	3	4	7	10	17	26
$\mathcal{O}(T_n)$	1	2	2	3	4	6	8	14	19

General $1 \times n$ Case

In the preceding example, knowing two $1 \times n$ cases allowed us to reduce the upper bound significantly without a large amount of extra effort. Here we give an expression for all $1 \times n$ rectangular tilings, where the tiles in T are allowed to have multiple colors.

Notation

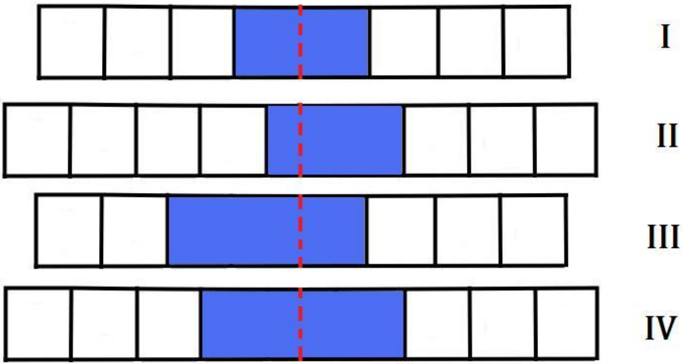
We begin by defining some convenient notation. Since we are covering boards of dimension $\{1 \times n | n \in \mathbb{N}\}$. Let $T = (a_1, a_2, a_3, \ldots)$, where a_m is the number of distinct colors of m–dominoes available. Then, T_n is the number of ways to tile a $1 \times n$ rectangle with colored dominoes in T. Connecting to our example, the Fibonacci numbers would be $T = (1, 1, 0, 0, 0, \ldots)$ while the Padovan numbers have $T = (0, 1, 1, 0, 0, 0, \ldots)$.

Coefficients

We also need to define a set of coefficients based on the parity of the domino length and the rectangle length.

$$c_j = \begin{cases} T_{n-\frac{j}{2}} & j \equiv n \equiv 0 \pmod 2 \\ 0 & j \equiv 0, n \equiv 1 \pmod 2 \\ 0 & j \equiv 1, n \equiv 0 \pmod 2 \\ T_{n-\frac{j-1}{2}} & j \equiv n \equiv 1 \pmod 2 \end{cases} \tag{4}$$

Coefficient Motivation



Complete Characterization of $1 \times n$ Tilings

Theorem

Let T be some set of colored k–dominoes, then the number of distinct tilings up to symmetry of a $1 \times n$ rectangle is equal to

$$\frac{1}{2}\left(T_n + \sum_{i=1}^{\infty} a_i c_i + \frac{T_{\frac{n}{2}}}{2} + \frac{(-1)^n T_{\frac{n}{2}}}{2}\right) \tag{5}$$

Lucas Tilings

It is natural to wonder if these methods could be adapted to give a similar formula for generalized Lucas tilings on a bracelet or necklace. Unfortunately, the complexity of the underlying symmetric group makes this a much more complex problem. Even in the simplest case we have:

Theorem

The number of distinct Lucas tilings of a $1 \times n$ bracelet up to symmetry is:

$$\sum_{i=0}^{\lceil \frac{n-1}{2} \rceil} \left[\frac{1}{n-i} \sum_{d|(i,n-1)} \varphi(d) \binom{\frac{n-i}{d}}{\frac{i}{d}} \right] \tag{6}$$

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Example 1

Theorem

The number of distinct rearrangements on a $2 \times n$ rectangle is

$$\frac{1}{4}\left(f_{2k}^2 + f_{2k} + 2f_k^2 + 2f_{k-1}^2\right) \tag{7}$$

when $n = 2k$ *and*

$$\frac{1}{4}\left(f_{2k+1}^2 + f_{2k+1} + 2f_k^2\right) \tag{8}$$

when $n = 2k + 1$

Example 2

Theorem

The number of distinct tilings of a $3 \times n$ *rectangle with squares of size* 1×1 *and* 2×2 *is*

$$\frac{1}{3}\left(2^{2n-1} + 2^n + 2^{n-1} + \frac{1+(-1)^n}{2}\right) \tag{9}$$

when n *is odd, and*

$$\frac{1}{3}\left(2^{2n} + 2^n + 2^{n-1} + 1\right) \tag{10}$$

when n *is even.*

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That's all...

Thank You.