Eugene Wigner [5]:

The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning.

Total Dynamics on Multiplex Networks
(or the unreasonable effectiveness of linear algebra)

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Abstract

Linear algebraic ideas occur in all branches of mathematics. In this talk, I will discuss some of the applications of these algebraic ideas in my research, focusing on recent work analyzing dynamical processes on multiplex networks.

Outline

Research Interests

- Enumerative Combinatorics
- Graph Theory
- LHCCRR
- Parallel Computing Algorithms
- Modular Forms
- Division by Three (Four, Nine Hundred and Twenty Two, etc.)
- Hodge Series
- Complex Networks

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- **LHCCRR**
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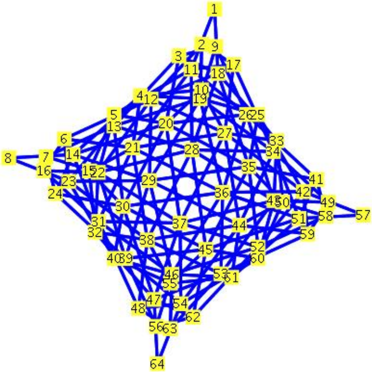
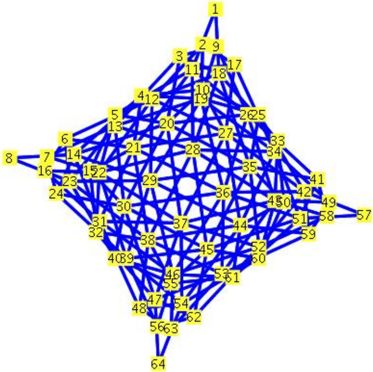
Graph Theory



Graph Theory



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LHCCRR

Èdouard Lucas:

> *The theory of recurrent sequences is an inexhaustible mine which contains all the properties of numbers; by calculating the successive terms of such sequences, decomposing them into their prime factors and seeking out by experimentation the laws of appearance and reproduction of the prime numbers, one can advance in a systematic manner the study of the properties of numbers and their application to all branches of mathematics.*

LHCCRR

$$0 \longrightarrow \operatorname{Tor}(\mathbb{C}^{\mathbb{N}}) \overset{i}{\hookrightarrow} \mathbb{C}^{\mathbb{N}} \overset{\pi}{\twoheadrightarrow} \mathbb{C}^{\mathbb{N}} / \operatorname{Tor}(\mathbb{C}^{\mathbb{N}}) \longrightarrow 0$$

$$[\mathbb{C}^{\mathbb{N}}]_f \cong \mathbb{C}^n$$

Modular Forms

Let Q be a quadratic form and P an associated spherical polynomial satisfying :

$$\sum_{1 \leq i,j \leq n} q^*_{i,j} \left(\frac{\partial^2 P}{\partial x_i \partial x_j} \right) \equiv 0.$$

We are interested in the map:

$$\varphi : \mathcal{P}(n,Q) \to \mathcal{M}(n,Q)$$

$$\varphi(P) = \theta(z;P,Q) = \sum_{v \in \mathbb{Z}^n} P(v) e^{2\pi i Q(v) z} = \sum_{m \in \mathbb{Z}} \left(\sum_{v \in \mathbb{Z}^n : Q(v) = m} P(v) \right) e^{2\pi i m z}$$

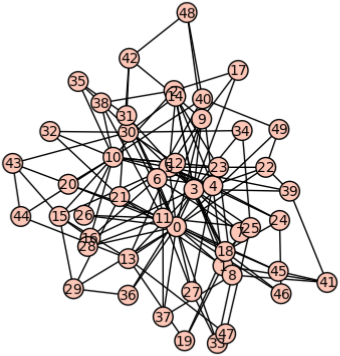
Hodge Series – Isospectral Manifolds

Let G be a finite subgroup of U_n.

$$\Lambda_G(x, y) = \frac{1}{|G|} \sum_{g \in G} \frac{\det(I_n + yg)}{\det(I_n - xg)}$$

Non–(almost)conjugate subgroups with identical Hodge Series give rise to Hodge isospectral orbifolds that are not strongly isospectral.

Complex Networks



Common Threads

Interesting aspects:

- Distinguished Basis
- Dimensionality
- Algebraic Invariants
- Linear Relations
- Module Perspectives

What are Complex Networks?

The idea at the heart of network theory is modeling real world systems with a (un)directed (un)weighted graph (network)[1].

[1]Must use different terminology to distinguish networks theorists from mathematicians :)

What are Complex Networks?

The idea at the heart of network theory is modeling real world systems with a (un)directed (un)weighted graph (network)[1].

Definition (Complex Network)

A “graph” with “non–trivial” “topological” “features”.

[1]Must use different terminology to distinguish networks theorists from mathematicians :)

Examples

- Internet
- World Wide Web
- Biological Networks
- Social Networks
- Economic Networks

Why is Networks not Graph Theory?

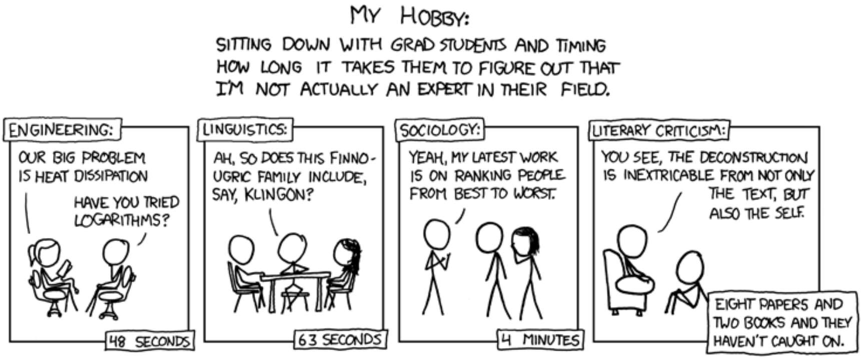
- Similarities:
    - Graphs
    - Underlying Linear Algebra

Why is Networks not Graph Theory?

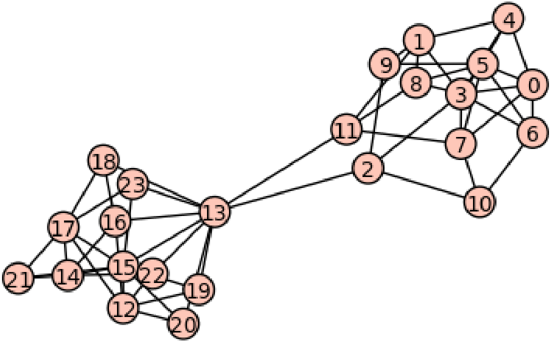
- Similarities:
  - Graphs
  - Underlying Linear Algebra

- Differences:
  - Historical positioning
  - Purposes
  - Specific topologies of interest (ER vs. AB)
  - Approximate vs. exact

Networks Basics (Centrality)



Networks Basics (Clustering)



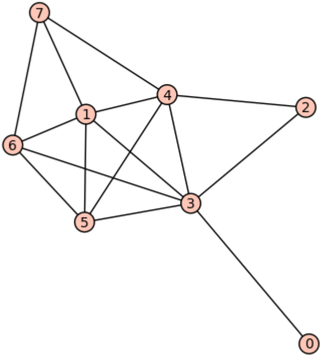
Functions on Spaces

A standard mathematical technique is to study spaces by studying the functions on those spaces. Examples include:

- Functional Analysis
- $\hom$ and categories
- Group characters and Group Actions

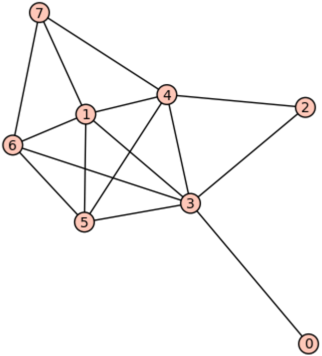
In the context of networks we can realize this idea by studying function $\varphi : V \to \mathbb{C}$. Since these maps are defined on a finite set, we can associate each φ with a vector $v_\varphi \in \mathbb{C}^n$ and we are firmly back in the land of linear operators.

Networks Basics (Degree Matrix)



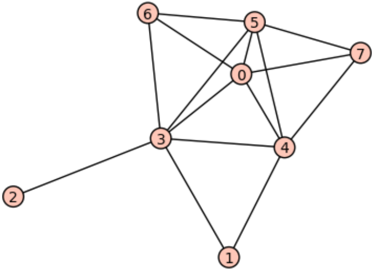
$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

Networks Basics (Adjacency Matrix)



$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Networks Basics (Incidence Matrix)



$$N = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

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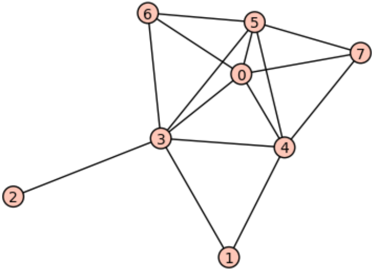
Networks Basics (Laplacian)



$$L = \begin{bmatrix} 5 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 2 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 6 & -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & -1 & 5 & -1 & 0 & -1 \\ -1 & 0 & 0 & -1 & -1 & 5 & -1 & -1 \\ -1 & 0 & 0 & -1 & 0 & -1 & 3 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & 3 \end{bmatrix}$$

Network Dynamics

As in the more theoretical subjects, we can get fine grained information about our networks of interested by considering various natural actions across the graph. Notions of centrality, clustering, and robustness can be addressed using these techniques. The process usually proceeds as follows:

1. Identify an objective function of interest
2. Use algebraic manipulations to discover an underlying matrix
3. Rephrase the function as an optimization problem over a single parameter or vector system
4. Use eigenvector analysis on the derived operator to solve the optimization problem

Eigenvector Centrality

In this case our objective function is a ranking of each node according to their importance. A natural way to accomplish this is to rank each node according to (a scalar multiple of) the sum of the ranks of its neighbors (knowing important people is important).

Given a ranking vector v, we then want to have the property that $v_i = \lambda \sum_{j \sim i} v_j$. The adjacency matrix A captures this information exactly, so we are really computing $v_i = \lambda \sum_j A_{i,j} v_j$. In vector terms this is just an eigenvalue problem: $v = \lambda A v$. We can then use the Perron–Frobenius Theorem to see that the solution we want is given by the leading eigenvector of A.

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This method can seem simple and contrived, but in fact Google uses a slight modification of this methodology to rank webpages for its search engine.

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Graph Laplacian

The graph Laplacian, defined as $L = D - A$, is perhaps the most useful matrix that can be associated to a network [1]. Various normalizations of this matrix such as $I - D^{-1}A$ are also quite useful. It has many useful algebraic properties as well as natural dynamical interpretations.

Incidence Matrices

The nice algebraic properties of the Laplacian, such as symmetry and positive semi–definiteness, follow directly from the construction of the Laplacian as NN^T, where N is the incidence matrix associated to a graph. The connectivity of the network is also captured by L, which can be seen by permuting N to place connected vertices in segments.

Clustering

The Laplacian also arises in the context of network clustering. Here we define the objective function to be a vector in $\{\pm 1\}^n$ that minimizes $\sum_{i,j} A_{i,j}(1 - v_i v_j)$, which captures the damage done to the network when the clusters are disconnected. Some algebraic manipulations (and relaxations) reduces this problem to minimizing the Rayleigh quotient: $\frac{v^T L v}{v^T v}$. This minimal meaningful solution is obtained by taking the second eigenvalue of L so the signs in the corresponding eigenvector partition the network efficiently.

Diffusion

The Laplacian also arises when we consider diffusion across a network. Given an initial vector φ we define the change at each vertex to be proportional to the difference in values at the end of each edge. This gives rise to a linear differential equation $\frac{d\varphi}{dt} = L\varphi$. The eigenvalues of L control the rate of diffusion across the network.

Multiplex Networks

A multiplex network is a collection of graphs all defined on the same edge set. Analyzing networks in this fashion allows us access to a greater amount of granularity in the data. At this point, dynamics across multiplex networks are poorly understood.

Examples

- Economic Networks
- Political Votes
- Social Networks
- Time–delay Networks

Dynamics on Multiplex Networks

There are two types of interactions that must be modeled, the exogenous connections captured by the edges on each layer, and the endogenous interactions that occur within the copies of each node. Connecting these dynamics in a principled fashion will give us an important tool for studying these (and all) networks in more detail.

Our Approach

Given a collection of dynamical operators, one for each level, we connect them by using a collection of scaled orthogonal projections to gather the data from each node and redistribute it across the copies. Combining these steps into a single operator gives us a tool to probe our network much like the various normalized Laplacians can be used for basic networks. The generality of this approach allows it to be applied even when the dynamical operators are not linear. As in all mathematics, the proof of value of a concept is in the new insights that it permits.

Matrix Realization

The matrix associated to the total operator takes on a convenient block diagonal form:

$$\begin{bmatrix} \alpha_{1,1}C_1D_1 & \alpha_{1,2}C_1D_2 & \cdots & \alpha_{1,k}C_1D_k \\ \alpha_{2,1}C_2D_1 & \alpha_{2,2}C_2D_2 & \cdots & \alpha_{2,k}C_2D_k \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{k,1}C_kD_1 & \alpha_{k,2}C_kD_2 & \cdots & \alpha_{k,k}C_kD_k \end{bmatrix}$$

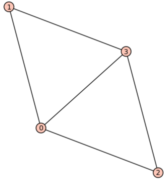
Where the $\{D_i\}$ are the dynamical operators associated to the layers and the $\{C_i\}$ are the diagonal proportionality matrices.

Preserved Properties

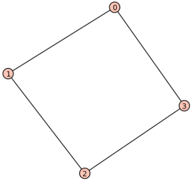
If the dynamics on each layer are assumed to have certain properties, we can prove that those properties are preserved in our operator:

- Irreducibility
- Primitivity
- Positive (negative) (semi)–definiteness
- Stochasticity

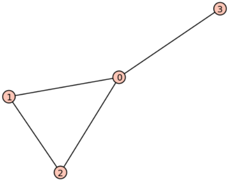
Multiplex Centrality



(a) Layer 1



(b) Layer 2



(c) Layer 3

Figure : A toy multiplex network

Multiplex Centrality (results)

Node	Level 1	Level 2	Level 3	$\hat{D}$
1	.5883	.5	.7071	.6438
2	.3922	.5	.4714	.4416
3	.3922	.5	.4714	.4190
4	.5883	.5	.2357	.4636

Table : Eigenvector centrality scores for the toy multiplex network

Eigenvalue Bounds For The Laplacian

The eigenvalues of the derived operator can be shown to be related to the eigenvalues of the sum of the individual operators. As mentioned previously, the eigenvalues of the Laplacian are perhaps the most important invariant of a graph.

- Fiedler Value: $\max_i(\lambda_f^i) \leq \lambda_f \leq \min_i(\lambda_1^i) + \sum_{j\neq \ell} \lambda_f^j$
- Leading Value: $\min_i(\lambda_1^i) \leq \lambda_1 \leq \sum_i \lambda_1^i$
- Synchronization: Directly computed as the quotient of the previous two bounds

Current Work

- More classes of operators
- Tighter eigenvalue bounds
- Non–linear dynamics
- Real world comparisons

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That's all...

Thank You!