Eugene Wigner [5]:

The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning.

More than just networks: Multiplex and Random Dot Product Structures

Daryl DeFord

Dartmouth College
Department of Mathematics

Department of Mathematics
Graduate Open House
April, 1 2015

Abstract

Complex network models are an important tool for understanding the world around us. However, many traditional network approaches are too coarse or rigid to allow for detailed analysis. In this talk I will present two approaches, multiplex models and random dot product models, that allow us to overcome some of these challenges.

Outline

What are Complex Networks?

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Definition (Complex Network)

???

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Definition (Complex Network)

???

- Graphs
- "Complex Structure"
- Generative or Adaptive Process

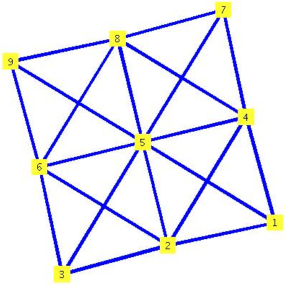
Why is Networks not Graph Theory?

- Similarities:
  - Graphs

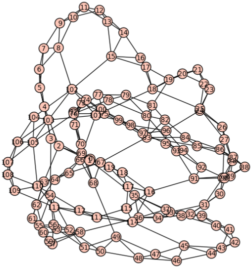
Why is Networks not Graph Theory?

- Similarities:
  - Graphs
- Differences:
  - Historical positioning
  - Purposes
  - Specific topologies of interest
  - Approximate vs. exact
  - ...

Examples

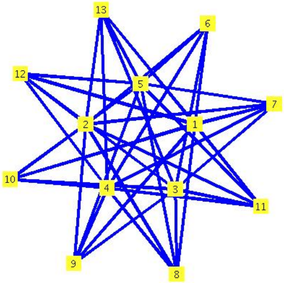


(a) Graph

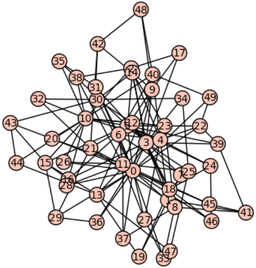


(b) Network

Examples

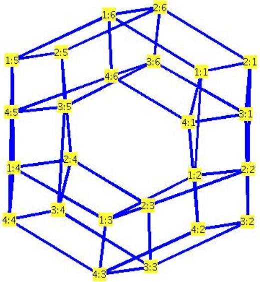


(c) Graph

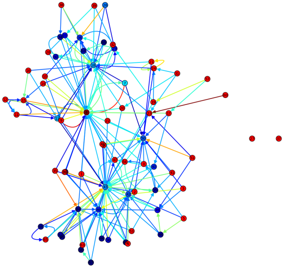


(d) Network

Examples



(e) Graph



(f) Network

Examples

- Transportation Networks
- Citation Networks
- Internet
- World Wide Web
- Biological Networks
- Social Networks
- Economic Networks
- ...

Examples

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- World Wide Web
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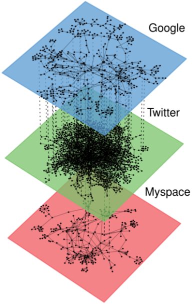
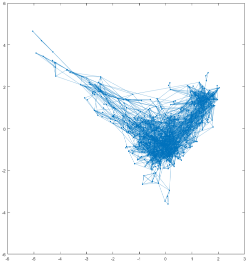
- **Social Networks**
- **Economic Networks**
- ...

Social Networks



(g) Online Social Networks[1]



(h) Karnataka Region Village 4

[1] Pymnet: http://www.plexmath.eu/?page-id=327

Economic Networks

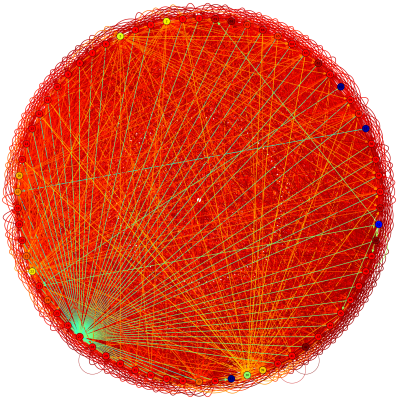


Figure : 2000 World Trade Web

Networks Basics (Centrality)

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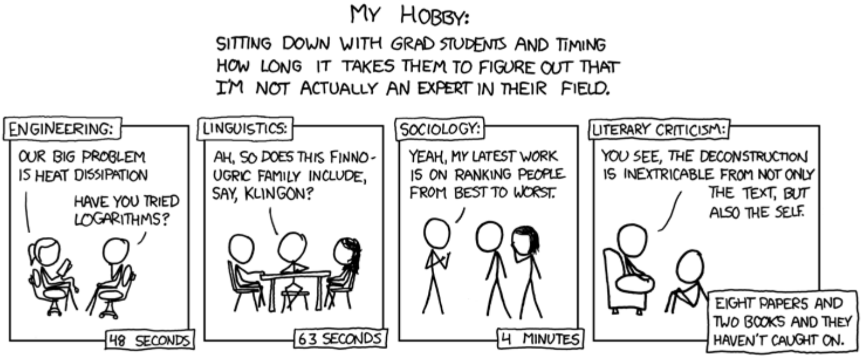

Figure : Impostor[1]

[1] https://xkcd.com/451/

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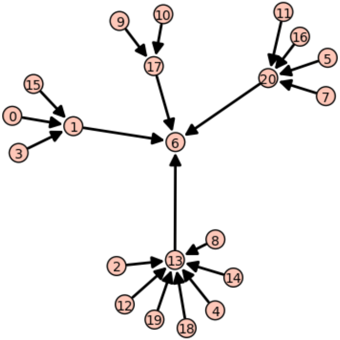

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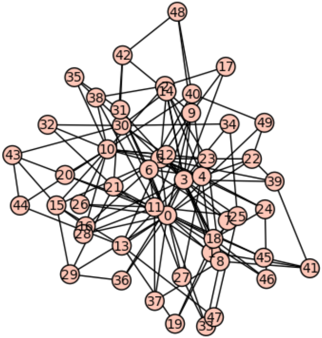

Networks Basics (Clustering)

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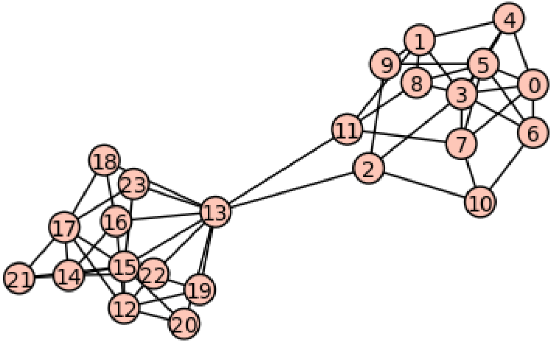

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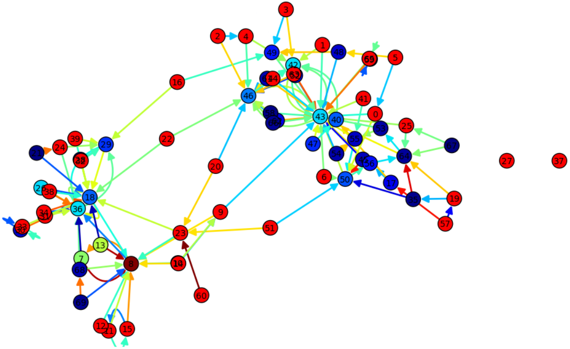
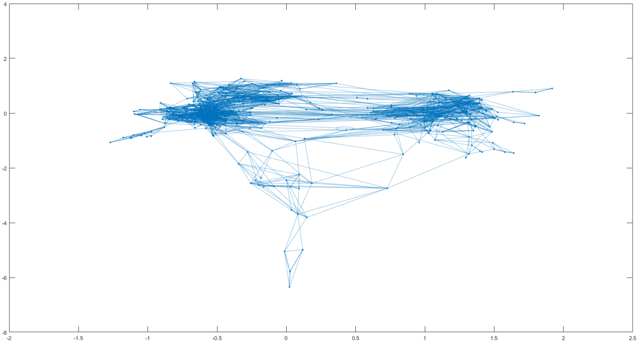
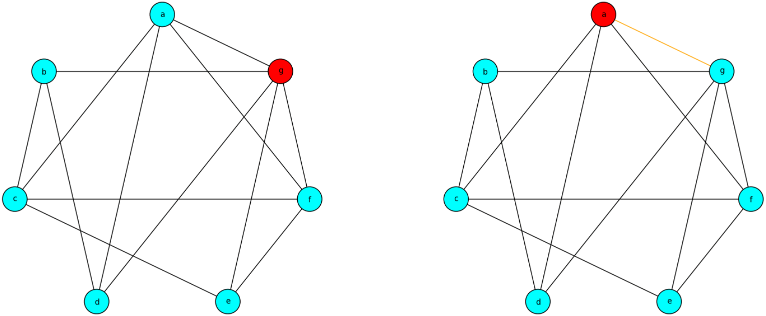


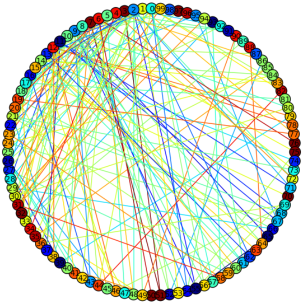
Networks Basics (Clustering)

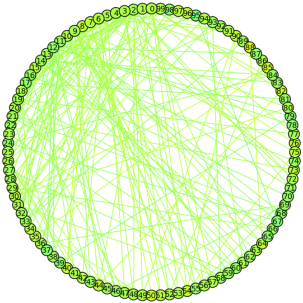


Random Walks



Diffusion





Multiplex Networks (Joint with S. Pauls)

Definition (Multiplex)

A multiplex network is a collection of individual networks (layers) all defined on the same edge set.

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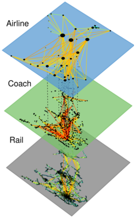
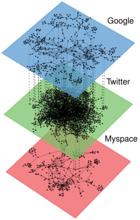


(g) Transportation[1]



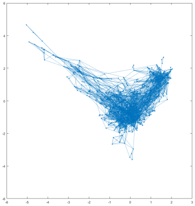
(h) Online Social[1]

[1] Pymnet: http://www.plexmath.eu/?page-id=327

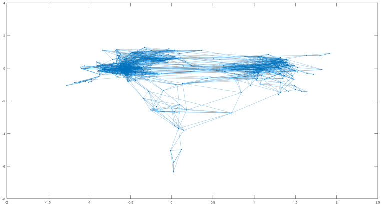
Social Networks

Layer Number	Relationship Type
0	Borrow money from
1	Give advice to
2	Help with a decision
3	Borrow kerosene or rice from
4	Lend kerosene or rice to
5	lend money to
6	Obtain medical advice from
7	Engage socially with
8	Are related to
9	Go to temple with
10	Invite to one's home
11	Visit in another's home

Full Village Networks

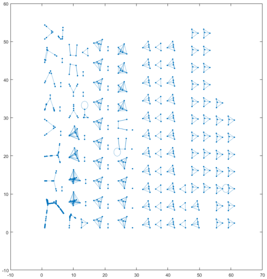


(i) Village 4

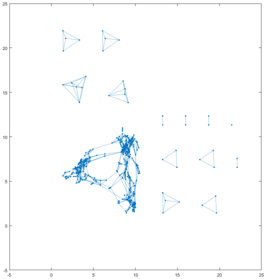


(j) Village 61

Medical Advice

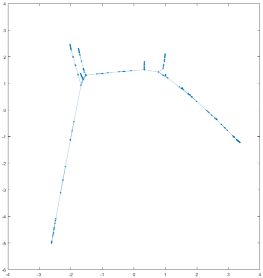


(k) Village 4

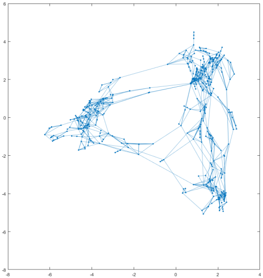


(l) Village 61

Medical Advice



(m) Village 4



(n) Village 61

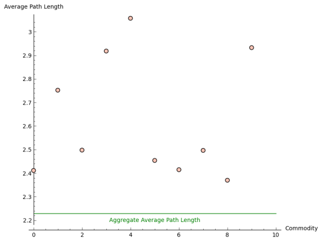
Dartmouth

WTW

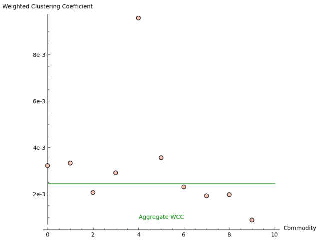
Layer	Description	Volume	% Total	Transitivity
0	Food and live animals	291554437	5.1	.82
1	Beverages and tobacco	48046852	0.9	.67
2	Crude materials	188946835	3.3	.79
3	Mineral fuels	565811660	10.0	.62
4	Animal and vegetable oils	14578671	0.3	.64
5	Chemicals	535703156	9.5	.83
6	Manufactured Goods	790582194	13.9	.87
7	Machinery	2387828874	42.1	.85
8	Miscellaneous manufacturing	736642890	13.0	.83
9	Other commodities	107685024	1.9	.56
All	Aggregate Trade	5667380593	100	.93

Table : Layer information for the 2000 World Trade Web.

Structural Distortions

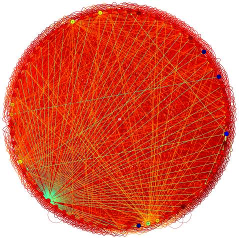


(o) Average Path Length

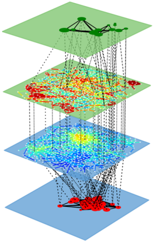


(p) Clustering Coefficient

Structural Models



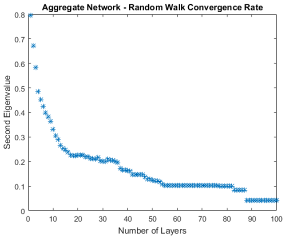
(q) Aggregate



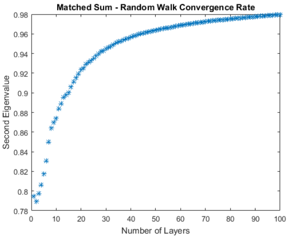
(r) Matched Sum[1]

[1] Pymnet: http://www.plexmath.eu/?page-id=327

Structural Distortions



(s) Aggregate



(t) Matched Sum

Dynamics on Multiplex Networks

- Two types of interactions
  - Within the individual layers
  - Between the layers
- Effects should "pass through" node copies
- Two step iterative model
- Interaction coefficients

$$(v')_i^{\alpha} = m_i^{\alpha,\beta} \sum_{\beta=1}^{k} c_i^{\alpha,\beta} (Dv)_i^{\beta}.$$

Matrix Realization

The matrix associated to the total operator takes on a convenient block diagonal form:

$$\begin{bmatrix} \alpha_{1,1}C_1D_1 & \alpha_{1,2}C_1D_2 & \cdots & \alpha_{1,k}C_1D_k \\ \alpha_{2,1}C_2D_1 & \alpha_{2,2}C_2D_2 & \cdots & \alpha_{2,k}C_2D_k \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{k,1}C_kD_1 & \alpha_{k,2}C_kD_2 & \cdots & \alpha_{k,k}C_kD_k \end{bmatrix}$$

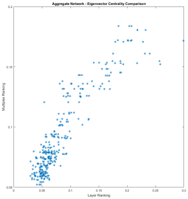
Where the $\{D_i\}$ are the dynamical operators associated to the layers and the $\{C_i\}$ are the diagonal proportionality matrices.

Preserved Properties

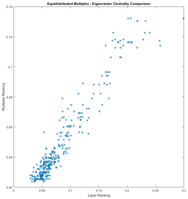
If the dynamics on each layer are assumed to have certain properties, we can prove that those properties are preserved in our operator:

- Irreducibility
- Primitivity
- Positive (negative) (semi)–definiteness
- Stochasticity

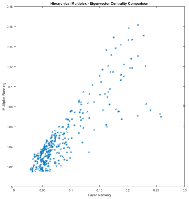
Centrality Comparison



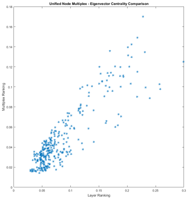
(u) Aggregate



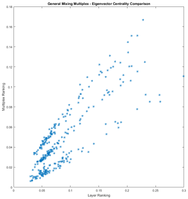
(v) Equidistribution



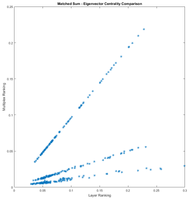
(w) Hierarchical Layer



(x) Unified Node



(y) General Mixing



(z) Matched Sum

Eigenvalue Bounds For The Laplacian

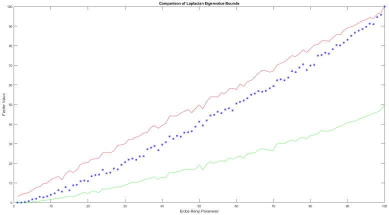
The eigenvalues of the derived operator can be shown to be related to the eigenvalues of the sum of the individual operators. As mentioned previously, the eigenvalues of the Laplacian are perhaps the most important invariant of a graph.

- Fiedler Value: $\max_\alpha(\lambda_F^\alpha) \leq k\lambda_F \leq \lambda_F^m + \sum_{\beta\neq m} \lambda_1^\beta$
- Leading Value: $\max_i(\lambda_1^i) \leq k\lambda_1 \leq \sum_i \lambda_1^i$

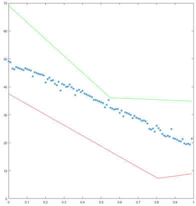
These bounds are special cases of the following more general but less computationally feasible bounds:

$$\max_i(\lambda_{n-j}^i) \leq k\lambda_{n-j} \leq \min_{J\vdash n+k-(j+1)} \left(\min_{\sigma\in S_n} \left(\sum_{\alpha=1}^{k} \lambda_{j_\alpha}^{\sigma(\alpha)} \right) \right)$$

Eigenvalue Bounds



(a) Equidistribution



(b) Hierarchical Layer Model

Null Models

Definition (Null Model)

A random network, parametrized to match some features of a given network, used to compare "expected" network measures.

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- Erdos–Renyi
- Configuration Model
- ERGMs

Random Dot Product Networks

- Generalization of several earlier models
- Intersection graphs
- Stochastic Block Models
- Interpretive Power
- Geometric tools

Random Dot Product Networks

- Generalization of several earlier models
- Intersection graphs
- Stochastic Block Models
- Interpretive Power
- Geometric tools

1. Draw n vectors from a distribution over $\mathbb{R}^d$
2. Place an edge between node i and node j with probability $\langle x_i, x_j \rangle$

Previous Results

- Generative Aspect
  - Diameter
  - Degree Distribution
  - Clustering Coefficient

Previous Results

- Generative Aspect
  - Diameter
  - Degree Distribution
  - Clustering Coefficient

- Inferential Aspect
  - Efficient Algorithm
  - Angle–based Clustering

Political Blogs

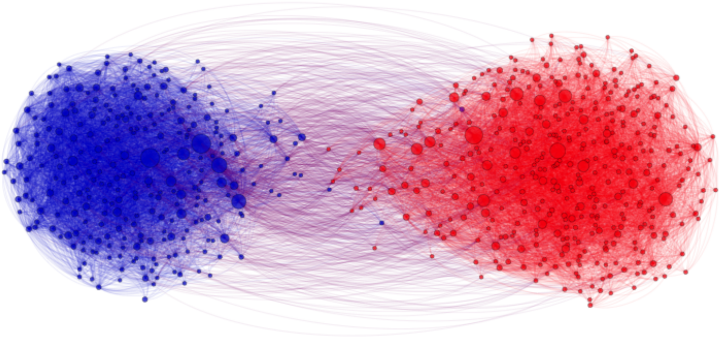


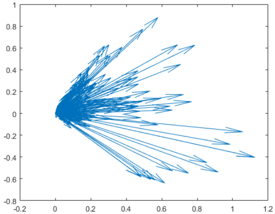
Figure : Visualization of political blogs network[2]

[2]http://allthingsgraphed.com/2014/10/09/visualizing-political-polarization/

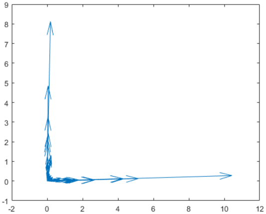
Generalizations (current work)

- Extend to weighted networks
- Replace the binomial distribution with an arbitrary parametrized probability distribution
- For each parameter, select a dimension and a distribution over vectors
- Much more freedom to adjust the model
- Generative and inferential aspects still work

Collaboration Networks



(a) Simple



(b) Weighted

That's all...

Thank You!

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