Combinatorial Rearrangements on Arbitrary Graphs

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Outline

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Overview

- Rearrangements on Graphs
- Rearrangements on Chessboards
- Tiling $m \times n$ Rectangles with Squares
- Symmetric Tilings

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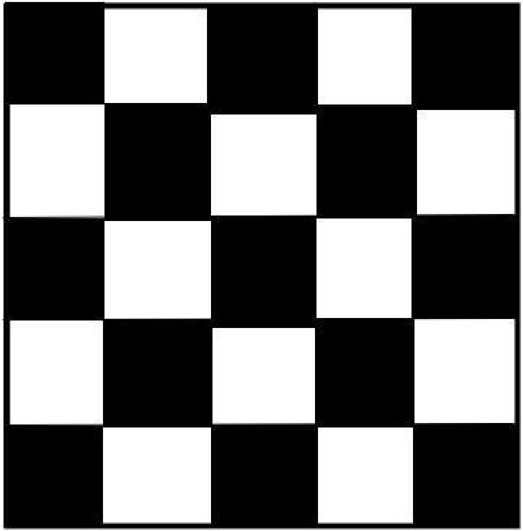
Motivation

Recurrence Relations

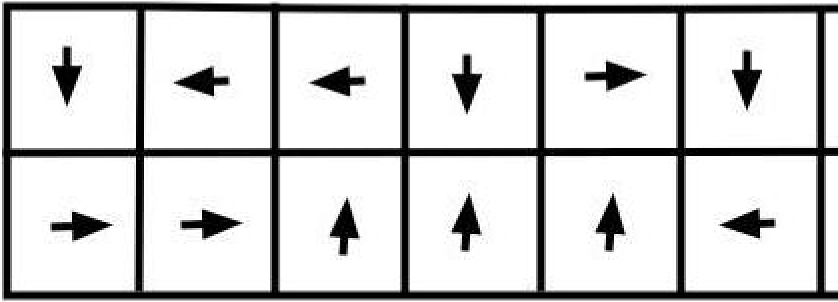
Original Problem (Honsberger)

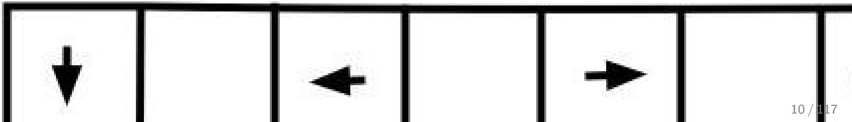
A classroom has 5 rows of 5 desks per row. The teacher requests each pupil to change his seat by going either to the seat in front, the one behind, the one to his left, or the one to his right (of course not all these options are possible to all students). Determine whether or not his directive can be carried out.

Original Problem



Seating Rearrangements and Tilings





Arbitrary Graphs

In order to count rearrangements on arbitrary graphs, we constructed the following problem statement:

Problem

Given a graph, place a marker on each vertex. We want to count the number of legitimate "rearrangements" of these markers subject to the following rules:

- *Each marker must move to an adjacent vertex.*
- *After all of the markers have moved, each vertex must contain exactly one marker.*

To permit markers to **either** *remain on their vertex or move to an adjacent vertex, add a self-loop to each vertex*

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Digraphs

With this problem statement we can describe these rearrangements mathematically as follows:

- Given a graph G, construct $\overleftrightarrow{G}$, by replacing each edge in G with a two directed edges (one in each orientation).
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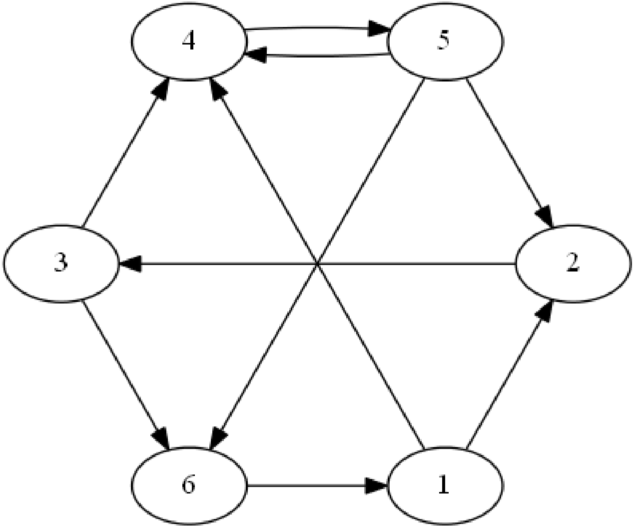
Cycle Covers

Definition

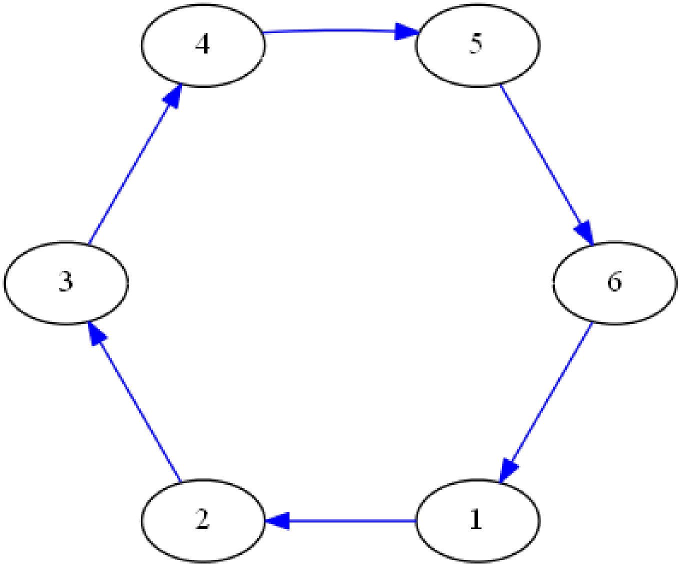
Given a digraph $D = (V, E)$, a cycle cover of D is a subset $C \subseteq E$, such that the induced digraph of C contains each vertex in V, and each vertex in the induced subgraph lies on exactly one cycle [7].

Permutation Parity

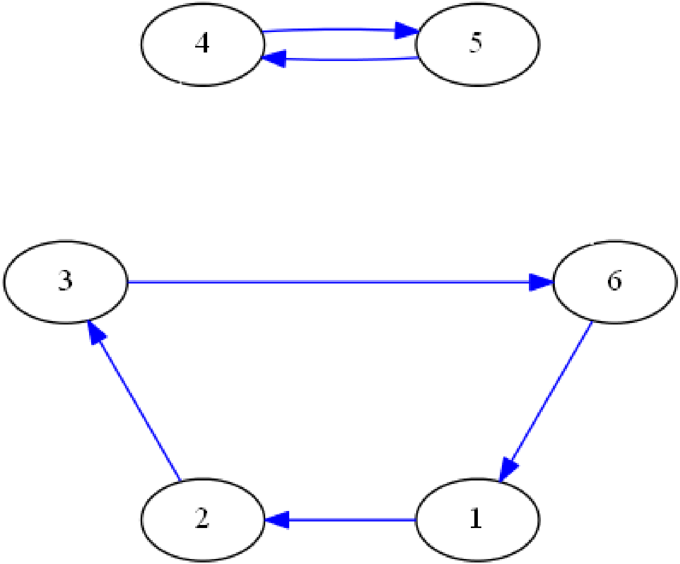
A cycle cover (permutation) is odd if it contains an odd number of even cycles.



Odd Cycle Cover



Even Cycle Cover



Permanents

The permanent of an $n \times n$ matrix, M, is defined as:

$$per(M) = \sum_{\pi \in S_n} \prod_{i=1}^{n} M_{i,\pi(i)},$$

- Determinant Similarities
- Differences
- Computational Complexity
- Counting with Permanents

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$Per(A) = Det(A)$

- Families of graphs with permanent equal to determinant
- Depends on the parities of the cycle cover
- $Per(A) = Det(A)$ iff A has no odd cycle covers [4]

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- Pòlya's question: Which matrices are convertible?
- Matrix Pfaffians [7]
- $Per(A) = Det(A')$ iff A has no subgraph homeomorhpic to $K_{3,3}$
- A' can be found in polynomial time (if it exists) [1]

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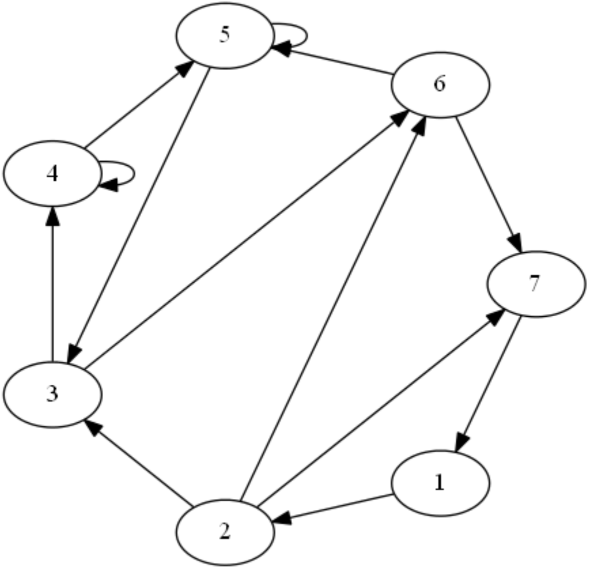
Labeled Digraph



Adjacency Matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$per(A) = 2$$

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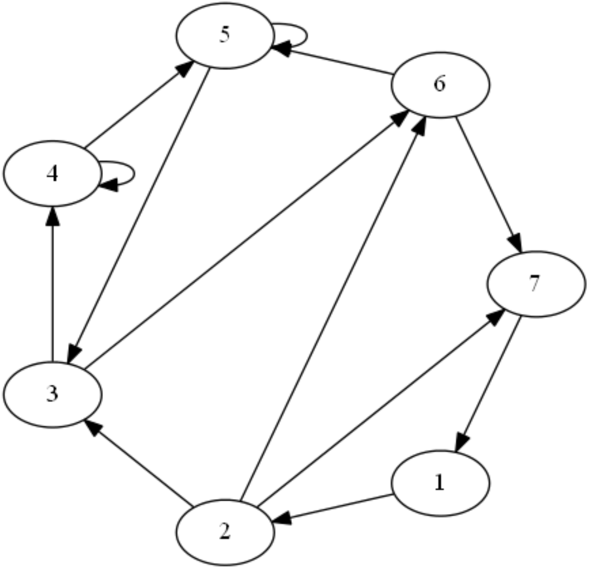
Cycle Covers



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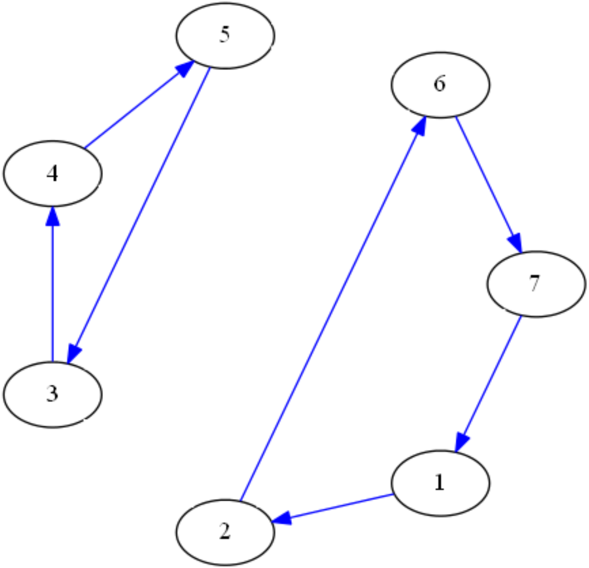

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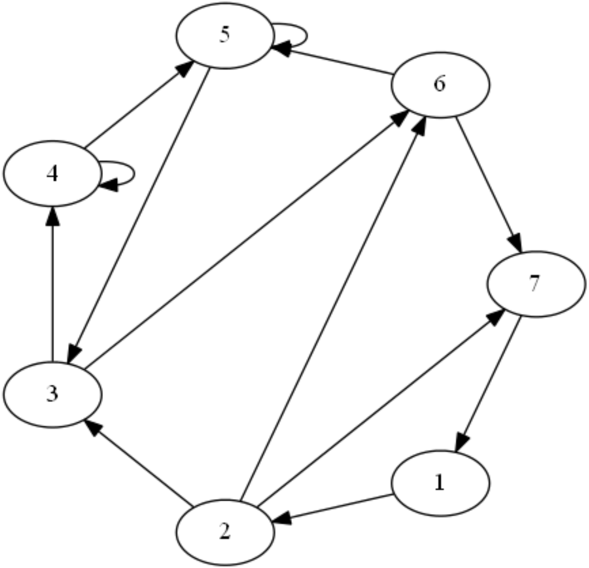

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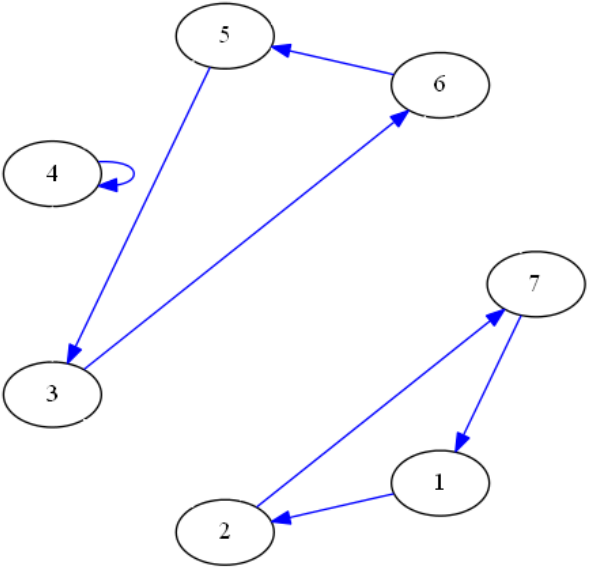

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Simple Graphs

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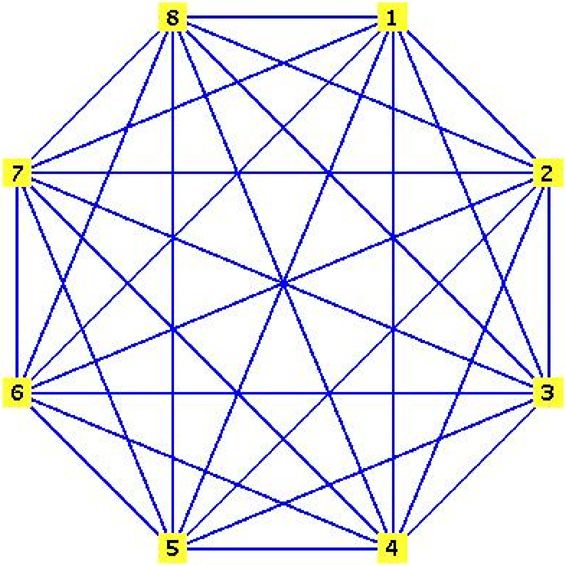
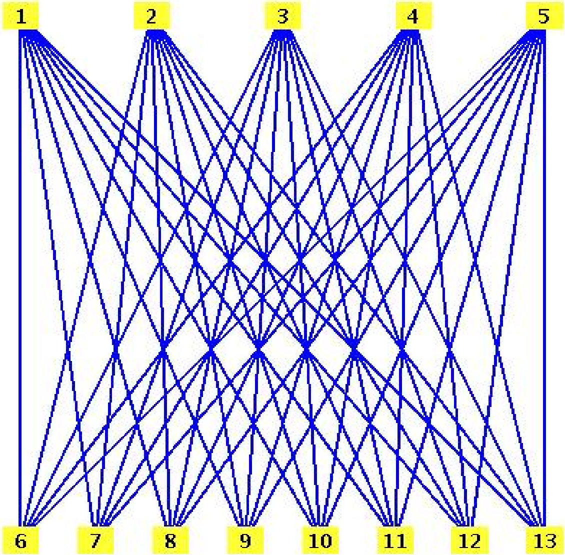
K_8



$K_{5,8}$



Simple Graphs

Graph	Rearrangements	With Stays
P_n	$0, 1, 0, 1, 0 \ldots$	f_n
C_n	$0, 1, 2, 4, 2, 4 \ldots$	$l_n + 2 = f_n + f_{n-2} + 2$
K_n	$D(n)$	$n!$
$K_{n,n}$	$(n!)^2$	$\sum_{i=0}^{n} [(n)_i]^2$
$K_{m,n}$ with $m \leq n$	0	$\sum_{i=0}^{m} (m)_i (n)_i$

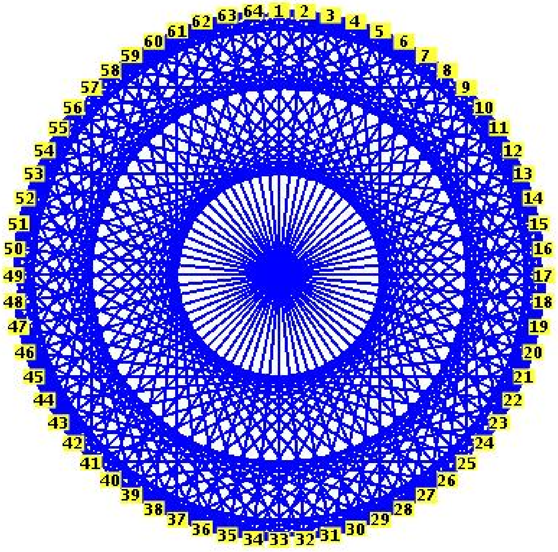
Computational Counting

Computational Counting

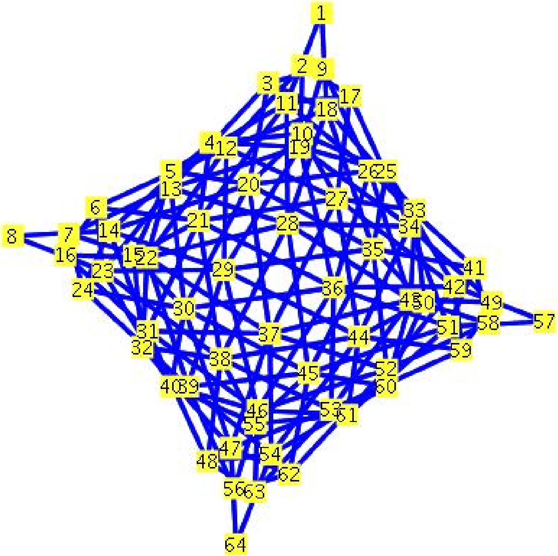
Game Pieces

Consider an $m \times n$ chessboard along with mn copies of a particular game piece, one on each square. In how many ways can the pieces be rearranged if they must each make one legal move? Or at most one legal move? Can these rearrangement problems be solved with recurrence techniques?

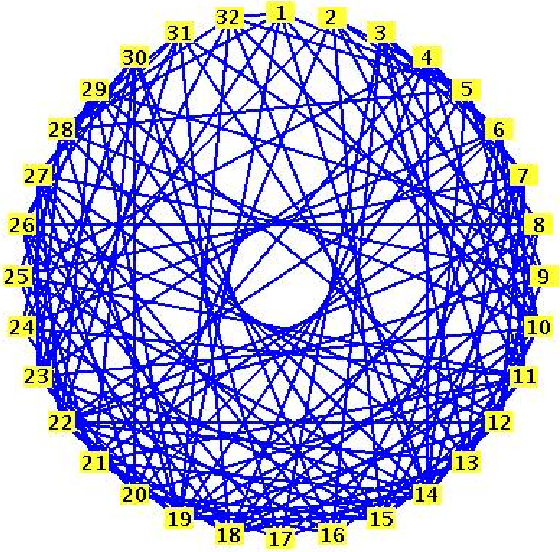
8 × 8 Rook Graph



8×8 Knight Graph



8 × 8 Bishop Graph



Fibonacci Relations

- $1 \times n$ Kings
- F_n
- $2 \times n$ Bishops
- F_n^2
- $2 \times 2n$ Knights
- F_n^4 or $F_n^2 * F_{n-1}^2$

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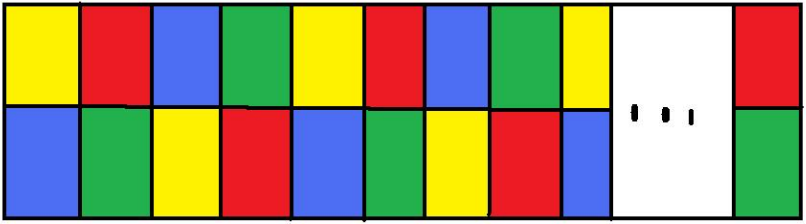
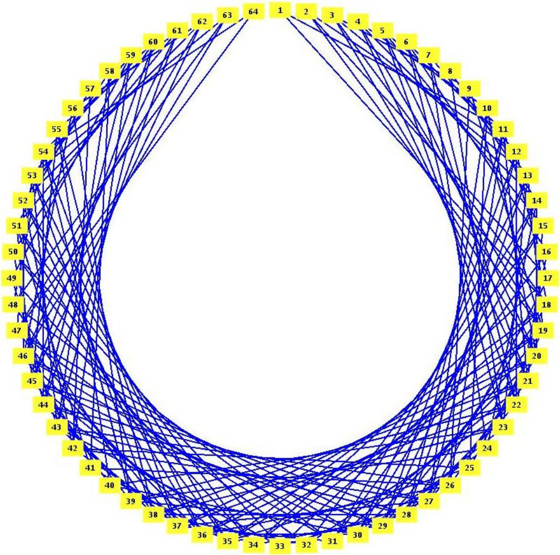
$2 \times 2n$ Knights



Knight Rearrangements



Knight's Tour

- 8×8 Knight's Tour
- 26,534,728,821,064 [4]
- 8×8 Knight Rearrangements
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Hosoya Index of Trees

The Hosoya index is a topological invariant from computational chemistry that is equivalent to the total number of matchings on a graph. This index correlates with many physical properties of organic compounds, especially the alkanes (saturated hydrocarbons).

Theorem

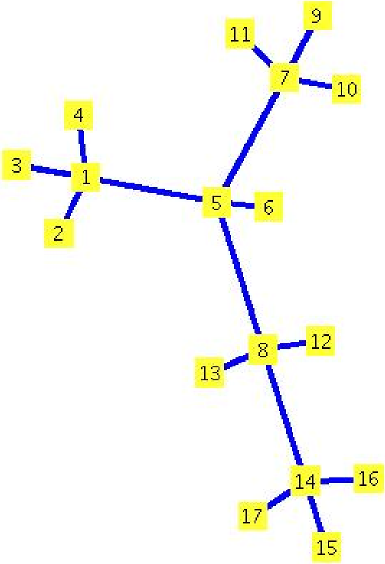
Let T be an $n-$tree with adjacency matrix $A(T)$. Then the Hosoya index of T is equal to $det(A(T)i + I_n)$

Hosoya Index Proof

Proof.

Sketch.
Since T is a tree there is a direct bijection between a given cycle cover on $\overleftrightarrow{T}$ with a self loop added to each vertex and a matching on T. Furthermore, $per(A(T) + I_n)$ counts these cycle covers. To see that $det(A(T)i + I_n) = per(A(T) + I_n)$ notice that each $2-$cycle and thus each even cycle counted in $det(A(T)i + I_n)$ has a weight of $i^2 = -1$, and thus that the weight of each cycle cover is equal to the sign of the permutation. □

Isopentane Example



$A(T)$

$$
per\left(\begin{bmatrix}
0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&1&1&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&1&1&1&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0
\end{bmatrix}\right) = 584
$$

$A(T)i + I_{17}$

$$
det\left(\begin{bmatrix}
1 & i & i & i & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
i & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
i & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
i & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 & 1 & i & i & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & i & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & i & 0 & 1 & 0 & i & i & i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & i & 0 & 0 & 1 & 0 & 0 & 0 & i & i & i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 1 & i & i & i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 1
\end{bmatrix}\right) = 584
$$

Theorems

Theorems

Bipartite Graphs Theorem

Theorem

Let $G = (\{U, V\}, E)$ be a bipartite graph. The number of rearrangements on G is equal to the square of the number of perfect matchings on G.

Bipartite Graphs Proof

Proof.

Sketch.
Construct a bijection between pairs of perfect matchings on G and cycle covers on $\overleftrightarrow{G}$. WLOG select two perfect matchings of G, m_1 and m_2. For each edge, (u_1, v_1) in m_1 place a directed edge in the cycle cover from u_1 to v_1. Similarly, for each edge, (u_2, v_2) in m_2 place a directed edge in the cycle cover from v_2 to u_2. Since m_1 and m_2 are perfect matchings, by construction, each vertex in the cycle cover has in–degree and out–degree equal to 1.

Given a cycle cover C on $\overleftrightarrow{G}$ construct two perfect matchings on G by taking the directed edges from vertices in U to vertices in V separately from the directed edges from V to U. Each of these sets of (undirected) edges corresponds to a perfect matching by the definition of cycle cover and the bijection is complete. □

$P_2 \times G$ Theorem

Theorem

The number of rearrangements on a bipartite graph G, when the markers on G are permitted to remain on their vertices, is equal to the number of perfect matchings on $P_2 \times G$.

$P_2 \times G$ Proof

Proof.

Sketch.
Observe that $P_2 \times G$ is equivalent to two identical copies of G where each vertex is connected to its copy by a single edge (P_2). To construct a bijection between these two sets of objects, associate a self–loop in a cycle cover with an edge between a vertex and its copy in the perfect matching. Since the graph is bipartite, the remaining cycles in the cycle cover can be decomposed into matching edges from U to V and from V to U as in the previous theorem.

□

Seating Rearrangements with Stays

- Applying the previous theorem to the original problem of seating rearrangements gives that the number of rearrangements in a $m \times n$ classroom, where the students are allowed to remain in place or move is equal to the number of perfect matchings in $P_2 \times P_m \times P_n$. These matchings are equivalent to tiling a $2 \times m \times n$ rectangular prism with $1 \times 1 \times 2$ tiles.
- A more direct proof of this equivalence can be given by identifying each possible move type; up/down, left/right, or stay, with a particular tile orientation in space.

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LHCCRR Theorem

Theorem

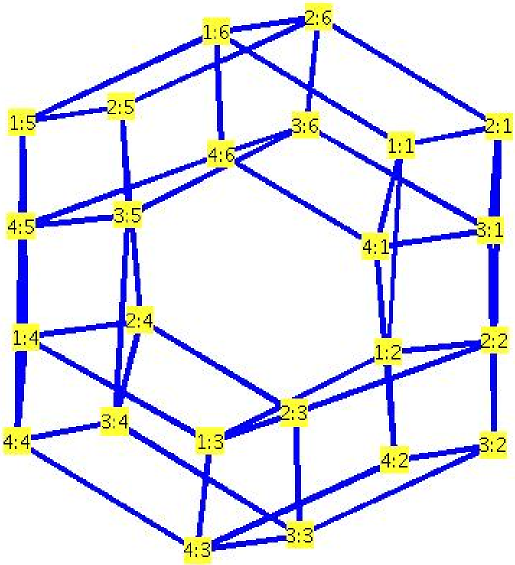
On any rectangular $m \times n$ board B with m fixed, and a marker on each square, where the set of permissible movements has a maximum horizontal displacement, the number of rearrangements on B satisfies a linear, homogeneous, constant–coefficient recurrence relation as n varies.

LHCCRR Proof

Proof.

Sketch.
Let d represent the maximum permissible horizontal displacement. Consider any set of marker movements that completes the first column. After all of the markers in the first column been moved, and other markers have been moved in to the first column to fill the remaining empty squares, any square in the initial $m \times d$ sub–rectangle may be in one of four states. Let S be the collection of all 4^{md} possible states of the initial $m \times d$ sub–rectangle, and let S^* represent the corresponding sequences counting the number of rearrangements of a board of length n beginning with each state as n varies. Finally, let a_n denote the sequence that describes the number of rearrangements on B as n varies.
For any board beginning with an element of S, consider all of possible sets of movements that "complete" the initial column. The resulting state is also in S, and has length $n - k$ for some k in $[1, d]$. Hence, the corresponding sequence can be expressed as a sum of elements in S^* with subscripts bounded below by $n - d$. This system of recurrences can be expressed as a linear, homogeneous, constant–coefficient recurrence relation in a_n either through the Cayley–Hamilton Theorem or by the successor operator matrix. □

Torus 4,6



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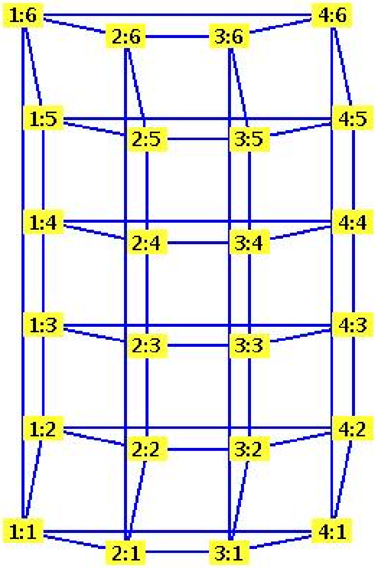

LHCCRR Extensions

- Cylinders
  - $C_m \times P_n$
  - $P_m \times C_n$
- Möbius strip
- Tori $C_m \times C_n$

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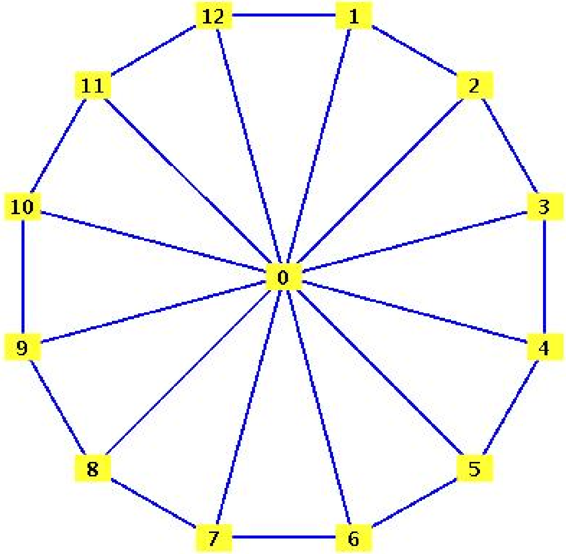
Wheel Graph Order 12



Wheel Graphs Rearrangements

The number of rearrangements on a wheel graph of order n is equal to n^2

- n odd
- Uniquely determined by the center vertex: $n \cdot n = n^2$
- n even
- Must create an odd cycle: $\frac{n}{2} \cdot 2n = n^2$

n	3	4	5	6	7	8	9	10	n
No stays	9	16	25	36	49	64	81	100	n^2
With stays	24	53	108	212	402	745	1356	2435	???

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Wheel Graph Rearrangements with Stays

The number of rearrangements on a wheel graph when the markers are permitted to either move or stay is equal to $nf_{n+2} + f_n + f_{n-2} - 2n + 2$.

Condition on the behavior of the center marker:

- if it remains in place,
- $C_n = f_n + f_{n-2} + 2$
- if it moves to one of the n other vertices,
- $nf_{n-1} + 2n\sum_{k=2}^{n} f_{n-k} = nf_{n-1} + 2nf_n - 2n$

$$\begin{aligned} f_n + f_{n-2} + 2 + nf_{n-1} + 2nf_n - 2n &= n((f_{n-1} + f_n) + f_n) + f_n + f_{n-2} - 2n + 2 \\ &= n(f_{n+1} + f_n) + f_n + f_{n-2} - 2n + 2 \\ &= nf_{n+2} + f_n + f_{n-2} - 2n + 2. \end{aligned}$$

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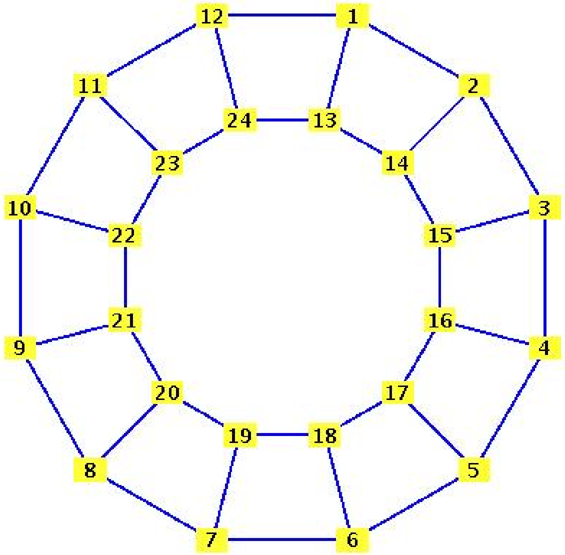
Prism Graph of Order 12



Prism Graphs

The number of rearrangements on a prism graph of order n is equal to $(l_n + 2)^2$ if n is even and $l_{2n} + 2$ if n is odd.

- n is even.
- The graph is bipartite and isomorphic to $C_n \times P_2$. Hence, the number of rearrangements is equal to the square of the number of rearrangements on C_n with stays permitted.
- n is odd.
- There is a bijection between pairs of Lucas tilings of length n and prism graph rearrangements where at least one marker moves between rows. The only uncounted rearrangements are the four where each marker remains in its original row. Thus, we have

$$l_n^2 + 4 = (l_n^2 + 2) = l_{2n} + 2$$

n	3	4	5	6	7	8	n
No stays	20	81	125	400	845	2401	$l_{2n} + 2$ | $(l_n + 2)^2$
With stays	82	272	890	3108	11042	39952	???

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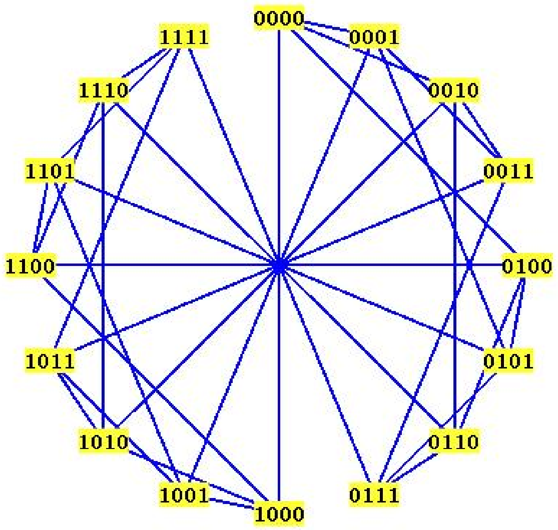
Hypercube of Order 4



Hypercubes

Since H_n is bipartite, the number of rearrangements on a $n-$cube is equal to the square of the number of perfect matchings on that cube. Similarly, because $H_n \cong H_{n-1} \times P_2$, the number of rearrangements with stays on a $n-$cube is equal to the number of perfect matchings in an $n+1$ cube.

	1	2	3	4	5
R(n) no stays	1	4	81	73984	347138964225
R(n) with stays	2	9	272	589185	16332454526976

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