Logistic Regression
Lecture 19

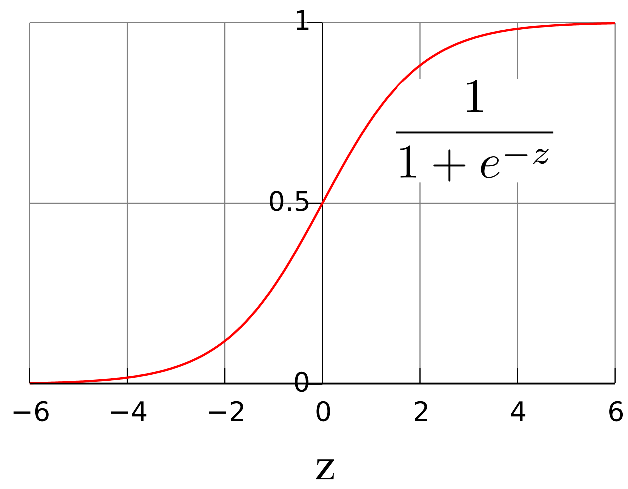
David Sontag
New York University

Slides adapted from Vibhav Gogate, Luke Zettlemoyer, Carlos Guestrin, and Dan Weld

Logistic Regression

- Learn P(Y|**X**) directly!
  - Assume a particular functional form
  - Sigmoid applied to a linear function of the data:

Logistic function (Sigmoid):



$$P(Y=1|X) = \frac{1}{1+\exp(w_0 + \sum_{i=1}^{n} w_i X_i)}$$

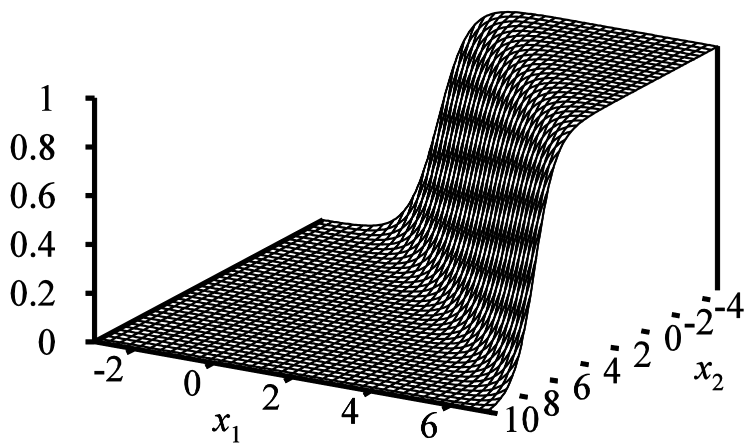
$$P(Y=0|X) = \frac{\exp(w_0 + \sum_{i=1}^{n} w_i X_i)}{1+\exp(w_0 + \sum_{i=1}^{n} w_i X_i)}$$

Features can be discrete or continuous!

Logistic Function in n Dimensions

$$P(Y=1|X) = \frac{1}{1+\exp(w_0 + \sum_{i=1}^{n} w_i X_i)}$$

Sigmoid applied to a linear function of the data:



Features can be discrete or continuous!

Logistic Regression: decision boundary

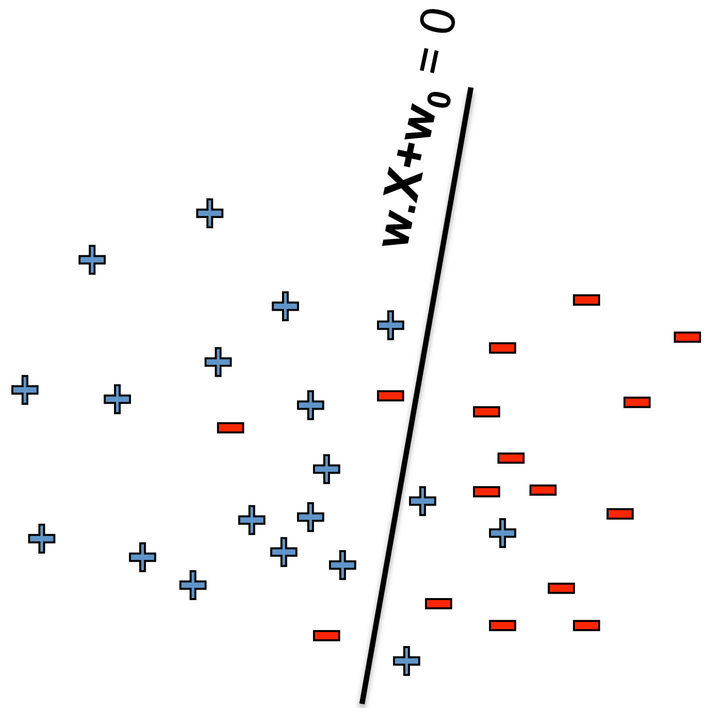
$$P(Y=1|X) = \frac{1}{1+\exp(w_0 + \sum_{i=1}^n w_i X_i)} \qquad P(Y=0|X) = \frac{\exp(w_0 + \sum_{i=1}^n w_i X_i)}{1+\exp(w_0 + \sum_{i=1}^n w_i X_i)}$$

- Prediction: Output the Y with highest P(Y|X)
  - For binary Y, output Y=0 if

$$1 < \frac{P(Y=0|X)}{P(Y=1|X)}$$

$$1 < \exp(w_0 + \sum_{i=1}^n w_i X_i)$$

$$0 < w_0 + \sum_{i=1}^n w_i X_i$$

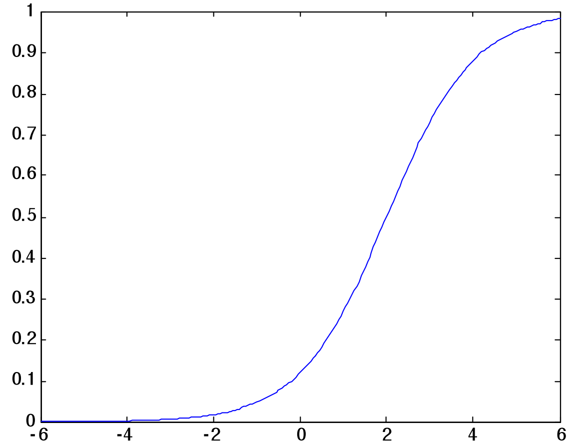


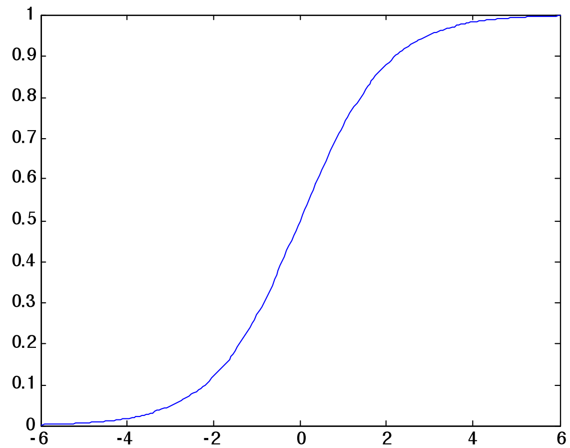
A Linear Classifier!

Understanding Sigmoids

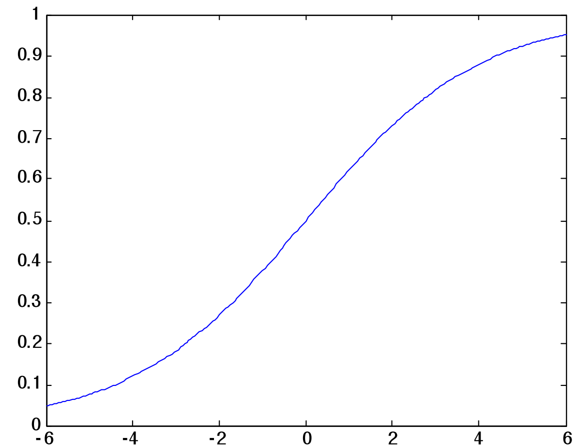
$$g(w_0 + \sum_i w_i x_i) = \frac{1}{1 + e^{w_0 + \sum_i w_i x_i}}$$

w_0=-2, w_1=-1





w_0=0, w_1=-1



w_0=0, w_1= -0.5

Likelihood vs. Conditional Likelihood

Generative (Naïve Bayes) maximizes **Data likelihood**

$$\begin{aligned} \ln P(\mathcal{D} \mid \mathbf{w}) &= \sum_{j=1}^{N} \ln P(\mathbf{x}^j, y^j \mid \mathbf{w}) \\ &= \sum_{j=1}^{N} \ln P(y^j \mid \mathbf{x}^j, \mathbf{w}) + \sum_{j=1}^{N} \ln P(\mathbf{x}^j \mid \mathbf{w}) \end{aligned}$$

Discriminative (Logistic Regr.) maximizes **Conditional Data Likelihood**

$$\ln P(\mathcal{D}_Y \mid \mathcal{D}_{\mathbf{X}}, \mathbf{w}) = \sum_{j=1}^{N} \ln P(y^j \mid \mathbf{x}^j, \mathbf{w})$$

Discriminative models *can't* compute P($\mathbf{x}^j$|$\mathbf{w}$)!

Or, … "They don't *waste effort* learning P(**X**)"

Focus only on P(Y|**X**) - all that matters for classification

Maximizing Conditional Log Likelihood

$$P(Y=0|X,W) = \frac{1}{1+exp(w_0+\sum_i w_i X_i)}$$

$$P(Y=1|X,W) = \frac{exp(w_0+\sum_i w_i X_i)}{1+exp(w_0+\sum_i w_i X_i)}$$

$$\begin{aligned} l(\mathbf{w}) &\equiv \ln \prod_j P(y^j|\mathbf{x}^j,\mathbf{w}) \\ &= \sum_j y^j(w_0+\sum_i^n w_i x_i^j) - \ln(1+exp(w_0+\sum_i^n w_i x_i^j)) \end{aligned}$$

0 or 1!

Bad news: no closed-form solution to maximize *l*(**w**)

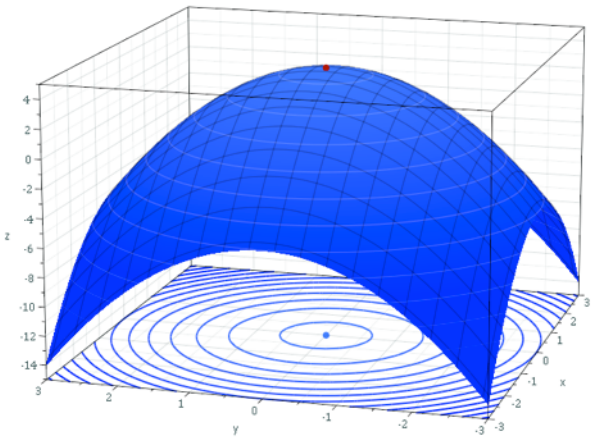
Good news: *l*(**w**) is concave function of **w**$\rightarrow$

No local minima

Concave functions easy to optimize

Optimizing concave function – Gradient ascent

- Conditional likelihood for Logistic Regression is concave $\rightarrow$



Gradient: $$\nabla_{\mathbf{w}} l(\mathbf{w}) = [\frac{\partial l(\mathbf{w})}{\partial w_0}, \ldots, \frac{\partial l(\mathbf{w})}{\partial w_n}]'$$

Learning rate, η>0

Update rule: $$\Delta \mathbf{w} = \eta \nabla_{\mathbf{w}} l(\mathbf{w})$$

$$w_i^{(t+1)} \leftarrow w_i^{(t)} + \eta \frac{\partial l(\mathbf{w})}{\partial w_i}$$

- Gradient ascent is simplest of optimization approaches

Maximize Conditional Log Likelihood: Gradient ascent

$$P(Y=1|X,W) = \frac{exp(w_0 + \sum_i w_i X_i)}{1 + exp(w_0 + \sum_i w_i X_i)}$$

$$l(\mathbf{w}) \;=\; \sum_j y^j (w_0 + \sum_i^n w_i x_i^j) - \ln(1 + exp(w_0 + \sum_i^n w_i x_i^j))$$

$$\frac{\partial l(w)}{\partial w_i} = \sum_j \left[\frac{\partial}{\partial w} y^j (w_0 + \sum_i w_i x_i^j) - \frac{\partial}{\partial w} \ln \left(1 + \exp(w_0 + \sum_i w_i x_i^j) \right) \right]$$

$$= \sum_j \left[y^j x_i^j - \frac{x_i^j \exp(w_0 + \sum_i w_i x_i^j)}{1 + \exp(w_0 + \sum_i w_i x_i^j)} \right]$$

$$= \sum_j x_i^j \left[y^j - \frac{\exp(w_0 + \sum_i w_i x_i^j)}{1 + \exp(w_0 + \sum_i w_i x_i^j)} \right]$$

$$\boxed{\frac{\partial l(w)}{\partial w_i} = \sum_j x_i^j \left(y^j - P(Y^j = 1 | x^j, w) \right)}$$

Gradient Ascent for LR

Gradient ascent algorithm: (learning rate η > 0)

do:

$$w_0^{(t+1)} \leftarrow w_0^{(t)} + \eta \sum_j [y^j - \hat{P}(Y^j = 1 \mid \mathbf{x}^j, \mathbf{w})]$$

For i=1 to n: (iterate over features)

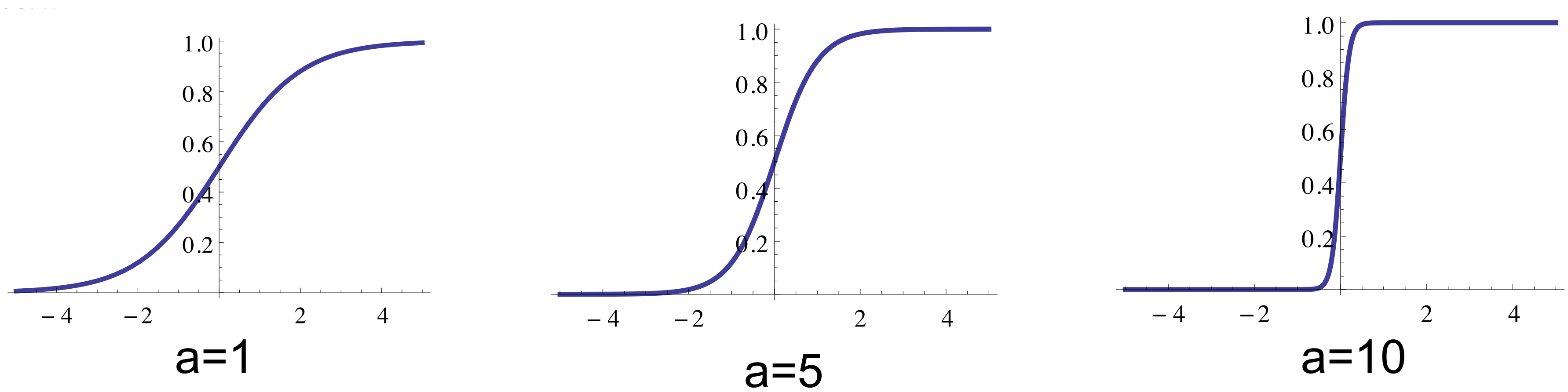
$$w_i^{(t+1)} \leftarrow w_i^{(t)} + \eta \sum_j x_i^j [y^j - \hat{P}(Y^j = 1 \mid \mathbf{x}^j, \mathbf{w})]$$

until "change" < ε

Loop over training examples!

Large parameters…

$$\frac{1}{1+e^{-ax}}$$



- Maximum likelihood solution: prefers higher weights
  - higher likelihood of (properly classified) examples close to decision boundary
  - larger influence of corresponding features on decision
  - *can cause overfitting!!!*
- Regularization: penalize high weights

That's all MLE. How about MAP?

$$p(\mathbf{w} \mid Y, \mathbf{X}) \;\; \propto \;\; P(Y \mid \mathbf{X}, \mathbf{w})p(\mathbf{w})$$

- One common approach is to define priors on **w**
  - Normal distribution, zero mean, identity covariance
  - "Pushes" parameters towards zero

$$p(\mathbf{w}) = \prod_i \frac{1}{\kappa\sqrt{2\pi}} \; e^{\frac{-w_i^2}{2\kappa^2}}$$

- ***Regularization***
  - Helps avoid very large weights and overfitting

- MAP estimate:

$$\mathbf{w}^* = \arg\max_{\mathbf{w}} \ln \left[p(\mathbf{w}) \prod_{j=1}^{N} P(y^j \mid \mathbf{x}^j, \mathbf{w}) \right]$$

MAP as Regularization

$$\mathbf{w}^* = \arg\max_{\mathbf{w}} \ln \left[p(\mathbf{w}) \prod_{j=1}^{N} P(y^j \mid \mathbf{x}^j, \mathbf{w}) \right] \quad p(\mathbf{w}) = \prod_i \frac{1}{\kappa\sqrt{2\pi}} \; e^{\frac{-w_i^2}{2\kappa^2}}$$

- Add log p(w) to objective:

$$\ln p(w) \propto -\frac{\lambda}{2} \sum_i w_i^2 \qquad \frac{\partial \ln p(w)}{\partial w_i} = -\lambda w_i$$

  - Quadratic penalty: drives weights towards zero
  - Adds a negative linear term to the gradients

Penalizes high weights, just like we did with SVMs!

MLE vs. MAP

- Maximum conditional likelihood estimate

$$\mathbf{w}^* = \arg\max_{\mathbf{w}} \ln \left[\prod_{j=1}^{N} P(y^j \mid \mathbf{x}^j, \mathbf{w}) \right]$$

$$\boxed{w_i^{(t+1)} \leftarrow w_i^{(t)} + \eta \sum_j x_i^j [y^j - \hat{P}(Y^j = 1 \mid \mathbf{x}^j, \mathbf{w})]}$$

- Maximum conditional a posteriori estimate

$$\mathbf{w}^* = \arg\max_{\mathbf{w}} \ln \left[p(\mathbf{w}) \prod_{j=1}^{N} P(y^j \mid \mathbf{x}^j, \mathbf{w}) \right]$$

$$\boxed{w_i^{(t+1)} \leftarrow w_i^{(t)} + \eta \left\{ -\lambda w_i^{(t)} + \sum_j x_i^j [y^j - \hat{P}(Y^j = 1 \mid \mathbf{x}^j, \mathbf{w})] \right\}}$$

Naïve Bayes *vs.* Logistic Regression

Learning: h:**X** $\mapsto$ Y

X – features

Y – target classes

Generative

- Assume functional form for
  - P(X|Y) **assume cond indep**
  - P(Y)
  - Est params from train data
- Gaussian NB for cont features
- Bayes rule to calc. P(Y|X= x)
  - P(Y | **X**) $\propto$ P(**X** | Y) P(Y)
- **Indirect** computation
  - Can also generate a sample of the data

Discriminative

- Assume functional form for
  - P(Y|X) **no assumptions**
  - Est params from training data
- Handles discrete & cont features
- Directly calculate P(Y|X=x)
  - Can't generate data sample

Naïve Bayes vs. Logistic Regression

[Ng & Jordan, 2002]

- Generative vs. Discriminative classifiers
- Asymptotic comparison

  (# training examples → infinity)
  - when model correct
    - NB and LDA (with class independent variances) and Logistic Regression produce identical classifiers
  - when model incorrect
    - LR is less biased – does not assume conditional independence
      - **therefore LR expected to outperform NB**

Naïve Bayes vs. Logistic Regression

[Ng & Jordan, 2002]

- Generative vs. Discriminative classifiers
- Non-asymptotic analysis
  - convergence rate of parameter estimates,

    (n = # of attributes in X)
    - Size of training data to get close to infinite data solution
    - Naïve Bayes needs $O(\log n)$ samples
    - Logistic Regression needs $O(n)$ samples

  - Naïve Bayes converges more quickly to its ***(perhaps less helpful)*** asymptotic estimates

Some experiments from UCI data sets

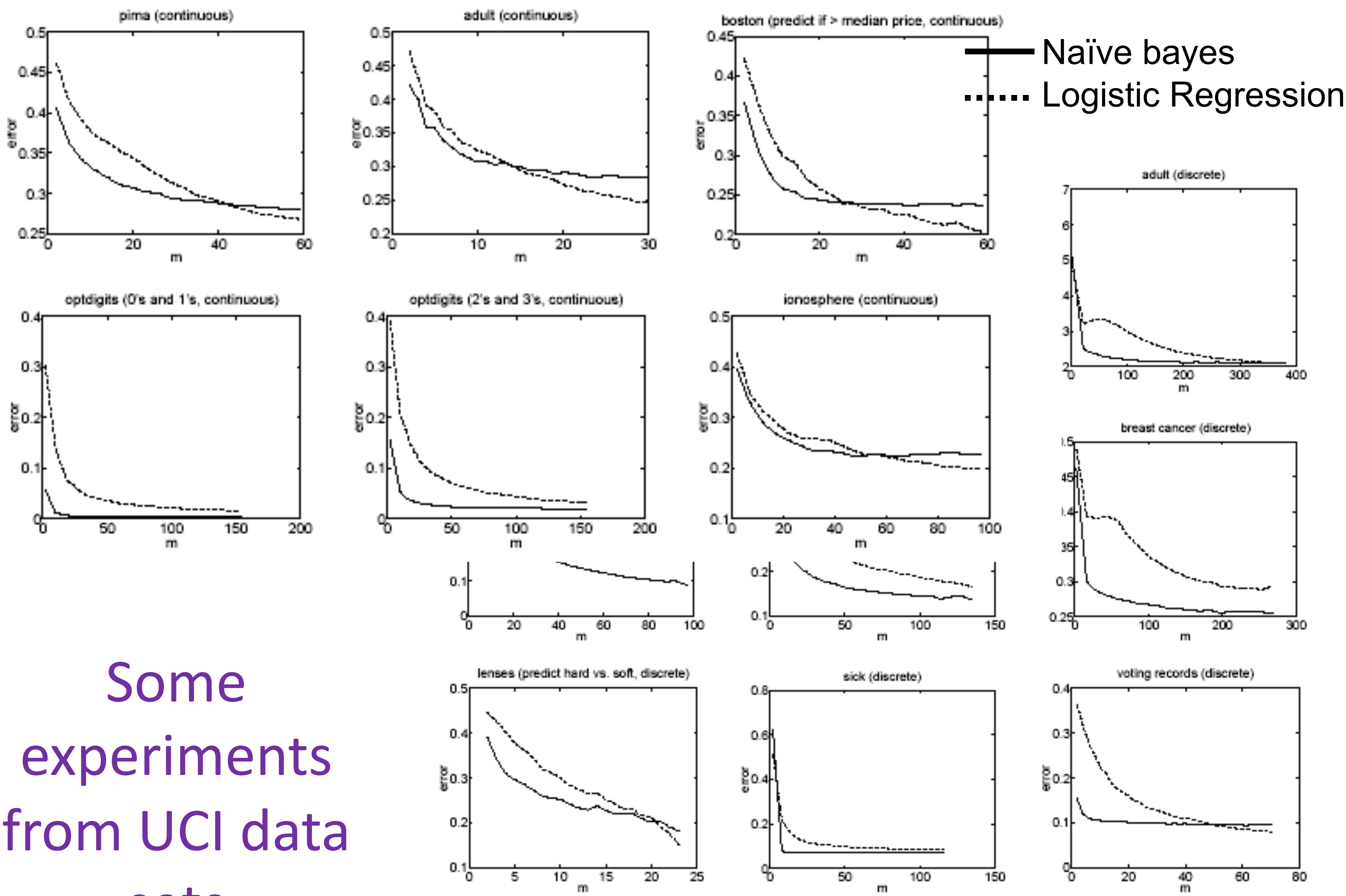


Figure 1: Results of 15 experiments on datasets from the UCI Machine Learning repository. Plots are of generalization error vs. m (averaged over 1000 random train/test splits). Dashed line is logistic regression; solid line is naive Bayes.

Logistic regression for discrete classification

Logistic regression in more general case, where set of possible *Y* is $\{y_1,\ldots,y_R\}$

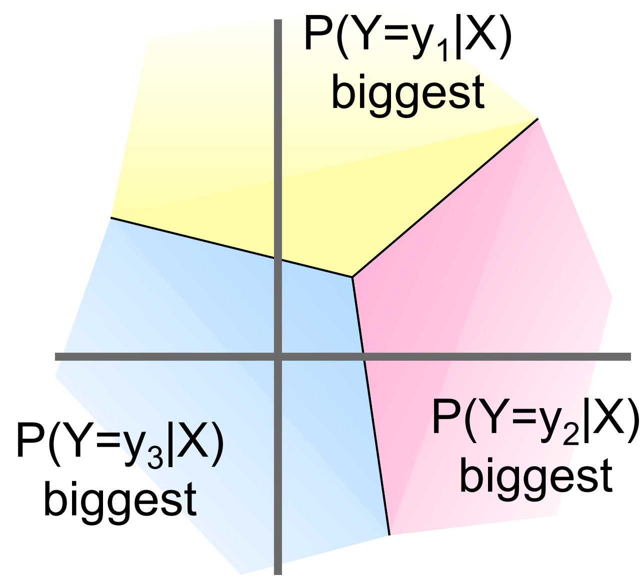
- Define a weight vector w_i for each y_i, i=1,…,R-1

$$P(Y=1|X) \propto \exp(w_{10} + \sum_i w_{1i} X_i)$$

$$P(Y=2|X) \propto \exp(w_{20} + \sum_i w_{2i} X_i)$$

$$\ldots$$

$$P(Y=r|X) = 1 - \sum_{j=1}^{r-1} P(Y=j|X)$$



Logistic regression for discrete classification

- Logistic regression in more general case, where
  Y is in the set $\{y_1,\ldots,y_R\}$

for $k<R$

$$P(Y = y_k|X) = \frac{\exp(w_{k0} + \sum_{i=1}^{n} w_{ki}X_i)}{1 + \sum_{j=1}^{R-1} \exp(w_{j0} + \sum_{i=1}^{n} w_{ji}X_i)}$$

for $k=R$ (normalization, so no weights for this class)

$$P(Y = y_R|X) = \frac{1}{1 + \sum_{j=1}^{R-1} \exp(w_{j0} + \sum_{i=1}^{n} w_{ji}X_i)}$$

Features can be discrete or continuous!