Introduction to Bayesian methods
Lecture 14

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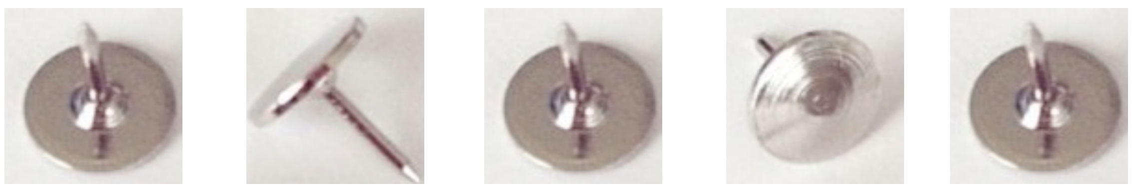
Slides adapted from Luke Zettlemoyer, Carlos Guestrin, Dan Klein, and Vibhav Gogate

Bayesian learning

- Bayesian learning uses **probability** to *model* data and *quantify uncertainty* of predictions
  - Facilitates incorporation of prior knowledge
  - Gives optimal predictions
    - Allows for decision-theoretic reasoning

Your first consulting job

- A billionaire from the suburbs of Manhattan asks you a question:
  - He says: I have thumbtack, if I flip it, what's the probability it will fall with the nail up?
  - You say: Please flip it a few times:



  - You say: The probability is:
    - P(heads) = 3/5
  - **He says: Why???**
  - You say: Because…

Outline of lectures

- Review of probability

(After midterm)

Maximum likelihood estimation

2 examples of Bayesian classifiers:

- Naïve Bayes
- Logistic regression

Random Variables

- A random variable is some aspect of the world about which we (may) have uncertainty
  - R = Is it raining?
  - D = How long will it take to drive to work?
  - L = Where am I?

- We denote random variables with capital letters

- Random variables have domains
  - R in {true, false}   (sometimes write as {+r, ¬r})
  - D in [0, ∞)
  - L in possible locations, maybe {(0,0), (0,1), …}

Probability Distributions

- Discrete random variables have distributions

$P(T)$

T	P
warm	0.5
cold	0.5

$P(W)$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

- A discrete distribution is a TABLE of probabilities of values
- The probability of a state (lower case) is a single number

$$P(W = rain) = 0.1 \qquad P(rain) = 0.1$$

- Must have:

$$\forall x\, P(x) \geq 0 \qquad \sum_x P(x) = 1$$

Joint Distributions

- A *joint distribution* over a set of random variables: $X_1, X_2, \ldots X_n$ specifies a real number for each assignment:

$$P(X_1 = x_1, X_2 = x_2, \ldots X_n = x_n)$$

$$P(x_1, x_2, \ldots x_n)$$

  - How many assignments if *n* variables with domain sizes *d*?
  - Must obey:

$$P(x_1, x_2, \ldots x_n) \geq 0$$

$$\sum_{(x_1, x_2, \ldots x_n)} P(x_1, x_2, \ldots x_n) = 1$$

$$P(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

- For all but the smallest distributions, impractical to write out or estimate
  - Instead, we make additional assumptions about the distribution

Marginal Distributions

- Marginal distributions are sub-tables which eliminate variables
- Marginalization (summing out): Combine collapsed rows by adding

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(t) = \sum_w P(t, w)$$

$P(T)$

T	P
hot	0.5
cold	0.5

$$P(w) = \sum_t P(t, w)$$

$P(W)$

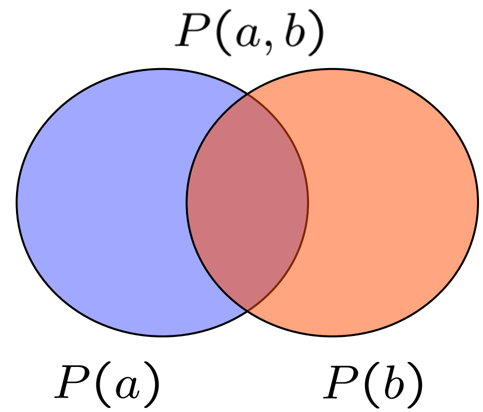
W	P
sun	0.6
rain	0.4

$$P(X_1 = x_1) = \sum_{x_2} P(X_1 = x_1, X_2 = x_2)$$

Conditional Probabilities

- A simple relation between joint and conditional probabilities
  - In fact, this is taken as the *definition* of a conditional probability

$$P(a|b) = \frac{P(a,b)}{P(b)}$$



$$P(T,W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W=r|T=c) = ???$$

Conditional Distributions

- Conditional distributions are probability distributions over some variables given fixed values of others

Conditional Distributions

$P(W|T)$

$$P(W|T = hot)$$

W	P
sun	0.8
rain	0.2

$$P(W|T = cold)$$

W	P
sun	0.4
rain	0.6

Joint Distribution

$$P(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

The Product Rule

- Sometimes have conditional distributions but want the joint

$$P(x|y) = \frac{P(x,y)}{P(y)} \iff P(x,y) = P(x|y)P(y)$$

- Example:

$P(W)$

W	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

$\iff$

$P(D,W)$

D	W	P
wet	sun	0.08
dry	sun	0.72
wet	rain	0.14
dry	rain	0.06

Bayes' Rule

- Two ways to factor a joint distribution over two variables:

$$P(x,y) = P(x|y)P(y) = P(y|x)P(x)$$

- Dividing, we get:

$$P(x|y) = \frac{P(y|x)}{P(y)}P(x)$$

- Why is this at all helpful?
  - Let's us build one conditional from its reverse
  - Often one conditional is tricky but the other one is simple
  - Foundation of many practical systems (e.g. ASR, MT)
- In the running for most important ML equation!