Advanced Algorithms 2020 Deadline: 11/06/2020, 11:59 pm

Graded Homework 1

Lecturer: Mohsen Ghaffari *Student Name:*

Note 1: *Solutions must be typeset in LaTeX and uploaded via moodle by 11:59 pm on November 6, 2020 (one pdf file for each problem). Late submissions will not be graded. If you would like to add a drawing in your solution, you can simply include a picture of a hand-drawn figure in your latex. You can submit only one file for each problem and the drawing should be incorporated in the rest of your explanation.*

Note 2: *This is a theory course and any claim should be substantiated with a proof. You can discuss the problems with the other students but you should write your own solutions independently, and you should be able to orally explain your submitted solution to the instructors, if asked. It is strictly prohibited to share any (hand)written or electronic (partial) solutions with fellow students. Moreover, if you discussed a problem with another student, you must list their names on your submitted solution.*

1 Destroying Cliques (12 points)

We are given an n-node undirected graph $G = (V, E)$ as input. The objective is to find a set $S \subset V$ of vertices, with the smallest possible cardinality $|S|$, such that after removing these vertices the remaining graph has no clique on 5 vertices. That is, in the induced subgraph $G[V \setminus S]$, there is no set of five distinct vertices that are all pairwise connected. Devise a deterministic constant approximation for this problem, with a time complexity in $\text{poly}(n)$.

2 Committee Formation (15 points)

We are given a list of n scientists $\{s_1, s_2, \ldots, s_n\}$, from whom we would like to assemble a committee. Recruiting the i^{th} scientist has cost $c_i \in \{1, \ldots, n^{10}\}$. The cost of a committee is simply the summation of the costs of its members. We are also given a list of m expertise $\{E_1, \ldots, E_m\}$. For each expertise E_i, the set of scientists in our list that have this expertise is $S_i \subset \{s_1, s_2, \ldots, s_n\}$, which is given to us as input, and we know that $3 \leq |S_i| \leq 10$. We would like to assemble a minimum-cost committee such that for each expertise, we have at least 3 scientists in our committee who have that expertise. It is perfectly fine if one scientist is counted on for several expertise.

Devise a polynomial-time approximation algorithm for this problem with a constant approximation guarantee, and for the smallest approximation factor that you can achieve.

3 Partitioning (15 points)

We are given an n-node undirected graph $G = (V, E)$ as input, with the guarantee that each node has degree at least d. Devise a polynomial-time algorithm that partitions the vertices V into at most $k = O(\log n) \cdot \frac{n}{d}$ parts $V_1, V_2, \ldots, V_k$ such that for each $i \in \{1, \ldots, k\}$, for any two nodes $u, v \in V_i$, we have $dist_G(u, v) \leq 2$.

Clarification (9:00 am on 26.10.2020): For problem 3, your algorithm must be deterministic. A randomized algorithm that works with high probability would receive partial score.

4 Path Optimization (20 points)

We are given an n-node directed acyclic graph $G = (V, E)$ and source and destination nodes s and t in V. Traversing each edge $e \in E$ costs $c_e \in \mathbb{Z}^+$ dollars and takes $t_e \in \mathbb{Z}^+$ minutes. Devise a fully polynomial-time approximation scheme for the cheapest way of (i.e., path) going from source s to destination t, subject to the constraint of having to arrive within T minutes, for a time budget T that is given as input.

5 Counting Good Paths (24 points)

We are given an n-node undirected graph $G = (V, E)$, along with a source node s and a destination node t. Moreover, each edge $e \in E$ is given a list of colors $L_e \subseteq \{1, ..., Q\}$. A path P_{st} connecting s to t is called *good* if and only if it is a shortest path from s to t and moreover, there is at least one color $q \in \{1, ..., Q\}$ such that all the edges $e' \in P_{st}$ on this path have color q in their list $L_{e'}$.

Let N be the number of good shortest paths between s and t. Devise a fully polynomial randomized approximation scheme for approximating N. That is, your algorithm should output an estimate $\hat{N}$ such up that $Pr[|\hat{N} - N] \geq \varepsilon N] \leq \delta$, for two given values $\varepsilon \in (0, 0.1)$ and $\delta \in (0, 0.1)$.

Hint *For a fixed color q, let S_q be the set of all the shortest paths from s to t where each edge e' on the path has color q in its list $L_{e'}$. You can compute $|S_q|$ exactly in polynomial time (how?). The objective is to approximate $N = |\cup_q S_q|$.*

6 Tree Embedding For Planar Graphs (14 points)

We are given a black-box ball carving algorithm $\mathcal{A}$ for planar graphs with the following guarantee: For any radius R, if we run algorithm $\mathcal{A}$ on any planar undirected graph $G = (V, E)$, the output of $\mathcal{A}$ is a partitioning of the vertices V into parts V_1, V_2, ..., such that (1) For each i, each two vertices $v, u \in V_i$ have $dist_G(u, v) \leq R$ and (2) for any two vertices u, v, the probability that u and v are in two different parts, say V_i and V_j for $j \neq i$, is at most $C \cdot \frac{dist_G(u,v)}{R}$, where C is some absolute constant C independent of $n = |V|$.

Using the above algorithm $\mathcal{A}$ as a black-box, prove that any unweighted planar graph with diameter at most D — i.e. where every two nodes have distance at most D — admits a probabilistic tree embedding with expected stretch at most $O(\log D)$. More formally, you should provide a polynomial-time randomized algorithm that generates an edge-weighted tree $T = (V, E_T)$ such that for any two vertices $v, u \in V$, we have

(P1) $dist_G(u, v) \leq dist_T(u, v)$, and

(P2) $\mathbb{E}[dist_T(u, v)] \leq dist_G(u, v) \cdot O(\log D)$.