Analytic gradient computation

A Layer-Based Restoration Framework for Variable-Aperture Photography

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Following the discussion in Appendix A, we present analytic formulas to compute the gradients of the objective function in their full form.

Multiple aperture settings The generalization to the multiple aperture settings is straightforward. We add an outer summation over aperture, include the exposure factors e_a, and relate the blur diameters over aperture setting according to the scale factors $s_a = \sqrt{e_a/e_A}$. For more details, see Eq. (3) and the discussion in footnote 2.

Inpainting To extend the gradient expressions to include inpainting, we denote the inpainting operator for layer k as $\mathcal{I}_k[\cdot]$, such that

$$\mathcal{I}_k[\,\mathbf{A}_k\mathbf{L}\,] = \mathbf{A}_k\mathbf{L} + \mathbf{A}_k^*\mathbf{L}_k^* \,. \tag{1}$$

We assume that $\mathcal{I}_k[\cdot]$ is a linear function of the radiance it takes as input, which covers a wide variety of inpainting schemes, including border replication, PDE-based diffusion, and exemplar-based inpainting.

In this way, the inpainting operator can be viewed as a large sparse matrix, $\boldsymbol{\mathcal{I}}_k$, left-multiplying the flattened scene radiance. Each column of $\boldsymbol{\mathcal{I}}_k$ thus describes how a particular layer pixel is diffused into the occluded regions.

Given this description, we can define the corresponding adjoint inpainting operator, $\mathcal{I}_k^\dagger[\cdot]$, which has the effect of "collecting" the diffused radiance from its final occluded destination and "returning" it to its unoccluded source. In terms of large matrix multiplication, the adjoint is simply the transpose $\boldsymbol{\mathcal{I}}_k^T$.

Full gradient expressions Putting these ideas together, we obtain:

$$\frac{\partial \mathcal{O}}{\partial \mathbf{L}} = -\sum_{a=1}^{A} e_a \left[\sum_{k=1}^{K} \mathcal{I}_k^\dagger \left[\boldsymbol{\Delta}_a \mathbf{A}_k \mathbf{M}_k \star B_{(s_a\sigma_k)} \right] \right] + \frac{\partial \|\mathbf{L}\|_\beta}{\partial \mathbf{L}} \tag{2}$$

$$\frac{\partial \mathcal{O}}{\partial \sigma_k} = -\sum_{a=1}^{A} s_a e_a \left[\sum_{x,y} \left[\sum_{k'=1}^{K} \mathcal{I}_k^\dagger \left[\boldsymbol{\Delta}_a \mathbf{A}_{k'} \mathbf{M}_{k'} \star \frac{\partial B_{(s_a\sigma_{k'})}}{\partial (s_a\sigma_{k'})} \right] \right] \mathbf{A}_k \mathbf{L} \right] , \tag{3}$$

where as before, $\star$ denotes 2D correlation, and for over-saturated pixels we revise these gradients to be zero. The gradient for the regularization term is unchanged:

$$\frac{\partial \|\mathbf{L}\|_\beta}{\partial \mathbf{L}} = -\operatorname{div}\left(\frac{w(\mathbf{L})^2\, \nabla\mathbf{L}}{\sqrt{\big(w(\mathbf{L})\,\|\nabla\mathbf{L}\|\big)^2 + \beta}} \right) . \tag{4}$$