Additional Proofs

Proof of Lemma 2. Let $\langle\ D,\ \tau,\ v\ \rangle$ be a tuple in $\mathcal{S}$, and let $[\alpha_1, \beta_1]$ be its depth of field. Now suppose that $\mathcal{S}$ contains another tuple whose depth of field, $[\alpha_2, \beta_2]$, overlaps with $[\alpha_1, \beta_1]$. Without loss of generality, assume that $\alpha_1 < \alpha_2 < \beta_1 < \beta_2$. We now replace $\langle\ D,\ \tau,\ v\ \rangle$ with a new tuple $\langle\ D',\ \tau',\ v'\ \rangle$ whose DOF is $[\alpha_1, \alpha_2]$ by setting D' according to Eq. (D) and v' according to Eq. (E). Since the DOF of the new tuple is narrower than the original, we have $D' > D$ and, hence, $\tau' < \tau$. Note that this tuple replacement preserves the synthetic DOF of the original sequence. We can apply this construction repeatedly until no tuples exist with overlapping DOFs. □

Proof of Lemma 3. From Eq. (5) it follows that the total exposure time is

$$\tau \ = \ \textstyle\sum_{i=1}^{n} \frac{L^*}{{D_i}^2}\ , \tag{16}$$

which is invariant to the permutation. To show that the synthetic DOF is also permutation invariant, we apply Eq. (12) recursively n times to obtain the right endpoint of the synthetic DOF:

$$\beta_n \ = \ \alpha \ \textstyle\prod_{i=1}^{n} \frac{D_i+c}{D_i-c}\ \ . \tag{17}$$

It follows that β_n is invariant to the permutation. □

Proof of Lemma 5. As in the proof of Lemma 4, we consider the case where $n = 2$. From that lemma it follows that the most efficient sequence involves splitting $[\alpha, \beta]$ into two sub-intervals $[\alpha, x]$ and $[x, \beta]$. To prove Lemma 5 we now show that the optimal split corresponds to a sequence with two identical aperture settings. Solving for $\frac{d\tau}{dx} = 0$ we obtain four solutions:

$$x = \left\{\ \pm\sqrt{\alpha\beta}\ ,\ \tfrac{(8\alpha\beta+\Delta)\ \pm\ (\beta-\alpha)\sqrt{\Delta}}{2(\beta+\alpha)}\ \right\}\ , \tag{18}$$

where $\Delta = \alpha^2 - 14\alpha\beta + \beta^2$. The inequality condition of Lemma 5 implies that $\Delta < 0$. Hence, the only real and positive solution is $x = \sqrt{\alpha\beta}$. From Eq. (D) it now follows that the intervals $[\alpha, \sqrt{\alpha\beta}]$ and $[\sqrt{\alpha\beta}, \beta]$ both correspond to an aperture equal to $c\ \frac{\sqrt{\beta}+\sqrt{\alpha}}{\sqrt{\beta}-\sqrt{\alpha}}$. To prove the Lemma for $n > 2$, we replace the sum in Eq. (14) with a sum of n terms corresponding to the subdivisions of $[\alpha, \beta]$, and then apply the above proof to each endpoint of that subdivision. This generates a set of relations, $\{\alpha_i = \sqrt{\alpha_{i-1}\alpha_{i+1}}\}_{i=2}^{n}$, which combine to define Eq. (11) uniquely.

Proof of Theorem 1. We first consider the most efficient capture sequence, $\mathcal{S}'$, among all sequences whose synthetic DOF is identical to $[\alpha, \beta]$. Lemmas 4 and 5 imply that the most efficient sequence (1) has maximal length and (2) uses the same aperture for all tuples. More specifically, consider such a sequence of n photos with diameter $D_i = D(n)$, for all i, according to Eq. (11). This sequence satisfies Eq. (17) with $\beta_n = \beta$, and we can manipulate this equation to obtain:

$$n \ = \ \frac{\log \frac{\alpha}{\beta}}{\log\left(\frac{D(n)-c}{D(n)+c}\right)}\ \ . \tag{19}$$

Note that while n increases monotonically with aperture diameter, the maximum aperture diameter D_{max} restricts the maximal n for which such an even subdivision is possible. This maximal n, whose formula is provided by Eq. (3), can be found by evaluating Eq. (19) with an aperture diameter of D_{max}.

While $\mathcal{S}'$ is the most efficient sequence among those whose synthetic DOFs equal to $[\alpha, \beta]$, there may be sequences whose DOF strictly contains this interval that are even more efficient. We now seek the most efficient sequence, $\mathcal{S}''$ among this class. To find it, we use two observations. First, $\mathcal{S}''$ must have length at most $n+1$. This is because longer sequences must include a tuple whose DOF lies entirely outside $[\alpha, \beta]$. Second, among all sequences of length $n+1$, the most efficient sequence is the one whose aperture diameters are all equal to the maximum possible value, D_{max}. This follows from the fact that any choice of $n+1$ apertures is sufficient to span the DOF, so the most efficient such choice involves the largest apertures possible.

From the above considerations it follows that the optimal capture sequence will be an equal-aperture sequence whose aperture will be either $D(n)$ or D_{max}. The test in Eq. (4) comes from comparing the total exposure times of the sequences $\mathcal{S}'$ and $\mathcal{S}''$ using Eq. (16). The theorem's inequality condition comes from Lemma 5. □

Proof of Theorem 2. The formulation of the integer linear program in Eqs. (7)–(9) follows in a straightforward fashion from our objective of minimizing total exposure time, plus the constraint that the apertures used in the optimal capture sequence must span the desired DOF.

First, note that the multiplicities n_i are non-negative integers, since they correspond to the number of photos taken with each discrete aperture D_i. This is expressed in Eq. (9). Second, we can rewrite the total exposure time given by Eq. (16) in terms of the multiplicities:

$$\tau \ = \ \sum_{i=1}^{m} n_i \ \frac{L^*}{{D_i}^2} \ , \tag{20}$$

This corresponds directly to Eq. (7), and is linear in the multiplicities being optimized. Finally, we can rewrite the expression for the right endpoint of the synthetic DOF provided by Eq. (17) in terms of the multiplicities as well:

$$\beta_m \ = \ \alpha \ \prod_{i=1}^{m} \left(\frac{D_i + c}{D_i - c} \right)^{n_i} \ . \tag{21}$$

Because all sequences we consider are sequential, the DOF $[\alpha, \beta]$ will be spanned without any gaps provided that the right endpoint satisfies $\beta_m \geq \beta$. By combining this constraint with Eq. (21) and taking logarithms, we obtain the inequality in Eq. (8), which also is linear in the multiplicities being optimized. □