The Lamb-Oseen Vortex and Paint Marbling

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Abstract

The displacement pattern arising from the decay of a two-dimensional Lamb-Oseen vortex in a Newtonian fluid can be closely modeled by the closed-form expression presented here. This formula enables Lamb-Oseen vortex simulation orders of magnitude faster than can be accomplished using finite-element methods, and without the accumulation of errors.

"French Curl" marbling patterns look as though they are created by a vortex. Analysis and simulation of a nineteenth century example of French Curl finds that the pattern was created without a vortex.

True vortexes are rarely seen in paint marbling because, in order to reach Reynolds numbers larger than 90, viscosity of the fluid bath must be much lower than is customarily used.

Keywords

Lamb-Oseen Vortex; paint marbling; Reynolds number; fuild mechanics

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1. Introduction

Marbling originated in Asia as a decorative art more than 800 years ago and has continued to evolve and thrive in parts of Asia and the Middle East. Marbling spread to Europe in the 1500s where it was employed for endpapers and book covers. Although Western interest declined with the advent of mechanized bookbinding, marbling has been enjoying a revival. It can even be seen in the heart-shaped designs drawn in latte foam.

Mathematically, it turns out that many common marbling techniques can be simply modeled by closed-form homeomorphisms[1]. A paper illustrating this approach is *Mathematical Marbling*[1]. The fluid mechanics of marbling are explored in *Oseen Flow in Paint Marbling*[2].

The previous work modeled creeping flows, ignoring momentum. The Lamb-Oseen vortex formula models the decay of the vortex over time, which includes momentum. Surprisingly, the Lamb-Oseen vortex is reversible due to its laminar nature.

[1] A homeomorphism is a continuous function between topological spaces (in this case between a topological space and itself) that has a continuous inverse function.

2. **Lamb-Oseen Vortex**

In two dimensions, the Lamb-Oseen vortex is a flow arising from an impulse of circulation at the center point. Its vorticity is nonzero only at the center point. The radial velocity is zero everywhere. The fluid moves in concentric shells around the center. The circumferential velocity is a continuous function of time t and radius r except at $r = 0$. Thus it is a laminar flow. It is also reversible; a clockwise impulse followed by a counterclockwise impulse of the same magnitude eventually returns to the original position.

Tryggeson[3] and Saffman[4] give the Lamb-Oseen formula for circumferential velocity of the irrotational vortex created by an impulse of circulation Γ (in m^2/s) at its center:

$$u_\theta = \frac{\Gamma}{2\,\pi\,r}\left[1 - \exp\left(-\frac{r^2}{4\,\nu\,t}\right)\right] \tag{1}$$

In the log-log plot of Figure 1, the dashed traces are the numerical integration of equation (1) with respect to time t; circulation $\Gamma = 10^{-3}\ \mathrm{m}^2/\mathrm{s}$. The three longest-dashed traces (radii 1 cm, 10 cm, 1 m) have slopes near one, indicating linear growth with time; the other trace is at 1 mm radius. The circumferential distance does not converge as $t \to \infty$. But a marbling tank is not left for hours to settle. Keeping t as a parameter to the formula, the only dimensionless combination of r, t, and ν from (1) is $r^2\,\nu^{-1}\,t^{-1}$. Combining this group with the constant 1 using the L^p-norm with the unusual value of $p = -3/4$ in equation (2) produces the values shown as squares in Figure 1.

$$p_\theta = \frac{\Gamma \cdot t}{2\,\pi\,r} \cdot \left[1 + \left(\frac{4\,\nu\,t}{2\,\pi\,r^2}\right)^{3/4}\right]^{-4/3} = \frac{\Gamma \cdot t}{2\,\pi\,r} \cdot \left\| 1, \frac{2\,\pi\,r^2}{4\,\nu\,t} \right\|_{-3/4} \tag{2}$$

where the L^p-norm is:

$$\|x, y\|_p = \left(|x|^p + |y|^p\right)^{1/p}$$

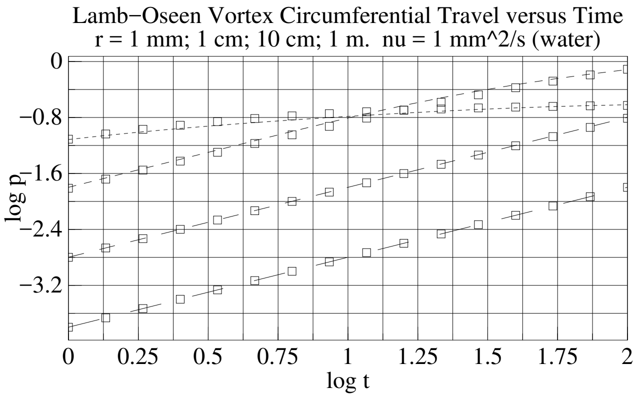


Figure 1

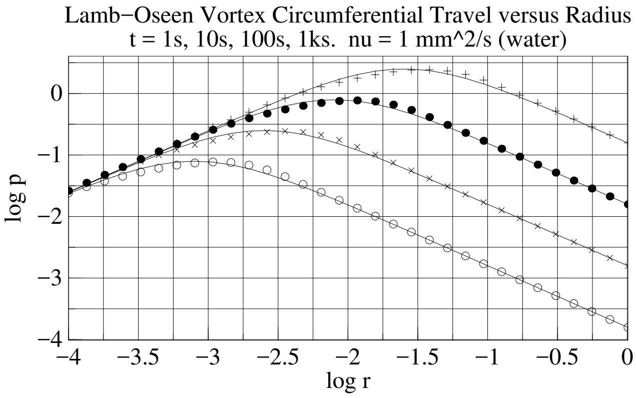


Figure 2

Figure 2 shows the travel versus radius r after 1 s (circles), 10 s (crosses), 100 s (dots), 1000 s (pluses); the lines are computed using formula (2).

The derivative of formula (2) with respect to t should be similar to the formula (1):

$$\frac{d\,p_\theta}{d\,t} = \frac{\Gamma}{2\,\pi\,r}\left[1+\left(\frac{4\,\nu\,t}{2\,\pi\,r^2}\right)^{3/4}\right]^{-7/3} \overset{?}{\approx} \frac{\Gamma}{2\,\pi\,r}\left[1-\exp\left(-\frac{r^2}{4\,\nu\,t}\right)\right] = u_\theta$$

The limit of both formulas as t approaches 0 is $\Gamma/(2\,\pi\,r)$; the limit of both formulas as t approaches ∞ is 0. Letting $4\,\nu\,r^{-2} = 1$ and $2\,\pi\,r/\Gamma = 1$, u_θ is compared with the $L^{-3/4}$-norm in Figure 3.

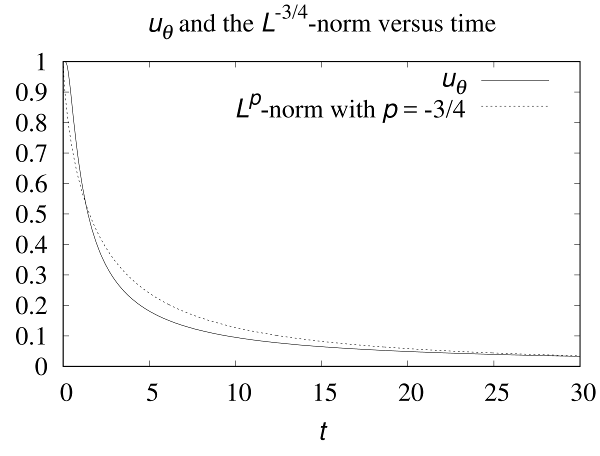


Figure 3

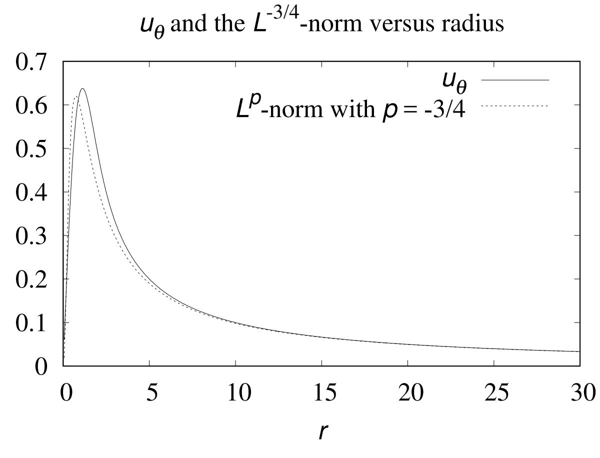


Figure 4

Letting $4\,\nu\,t = 1$ and $2\,\pi/\Gamma = 1$, u_θ is compared with the $L^{-3/4}$-norm in Figure 4.

3. **Angle**

For the marbling transform it is more convenient to work with angle a than circumferential distance p_θ. With the introduction of variable $\zeta = t/(2\,\pi\,r^2)$, the formula simplifies:

$$a = \frac{p_\theta}{r} = \Gamma\,\zeta\left[1 + (4\,\nu\,\zeta)^{3/4}\right]^{-4/3} = \Gamma\,\zeta\left\|1, \frac{1}{4\,\nu\,\zeta}\right\|_{-3/4} = \Gamma\left\|\zeta, \frac{1}{4\,\nu}\right\|_{-3/4} = \Gamma\left\|\frac{t}{2\,\pi\,r^2}, \frac{1}{4\,\nu}\right\|_{-3/4} \quad (3)$$

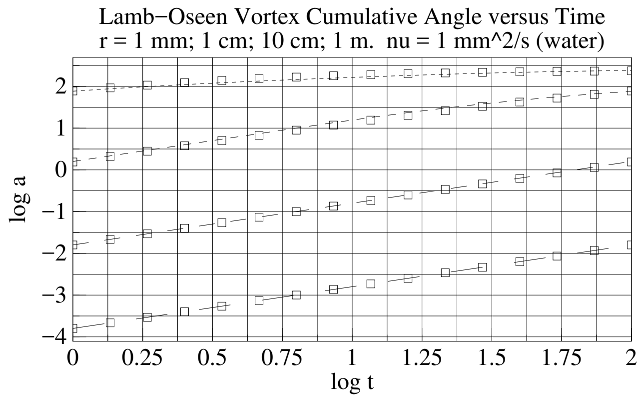


Figure 5

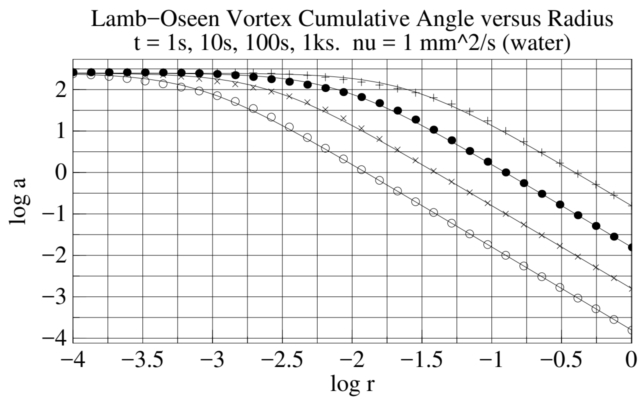


Figure 6

As time increases, the traces in Figure 5 eventually merge, resulting in the original pattern rigidly rotated.

A vortex homeomorphism of circulation Γ around $\vec{C}$ after time t maps a point $\vec{P}$ to:

$$\vec{C} + \left[\vec{P} - \vec{C}\right] \cdot \begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix} \qquad r = \left\|(\vec{P} - \vec{C})\right\|$$

Figure 7 shows a base pattern with no vortex. Figure 8 shows the pattern a short time after the impulse of circulation; the center of the vortex is rotated counterclockwise nearly 360°. After this the center rotates only a few degrees; the initial rotation propagates outward. Figure 9 shows the pattern after 10 times the duration as Figure 8. Figure 10 shows the pattern after 100 times the duration as Figure 8. After a very long time the pattern asymptotically returns to Figure 7.

On learning that the Lamb-Oseen vortex flow is reversible, Martin Jaffer immediately suggested that one could emulate sweeping freehand curves by applying the vortex homeomorphism, drawing a stylus on a straight path through it, then applying the reverse vortex homeomorphism, as shown in Figure 11. Applying the vortex homeomorphism to a half-black, half-white base creates a design evocative of the yin-yang pattern in Figure 12.

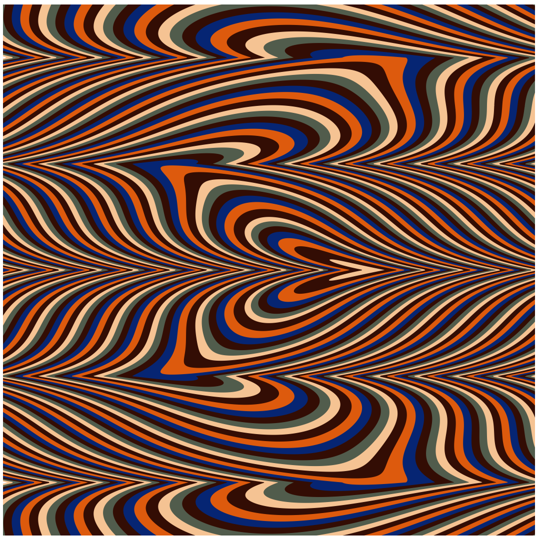
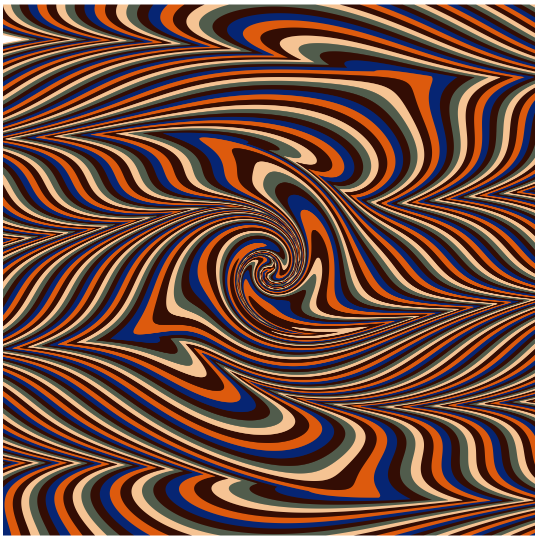


Figure 7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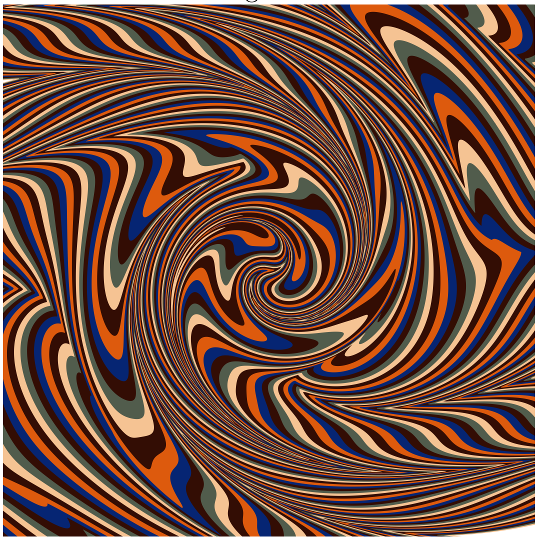
Figure 9

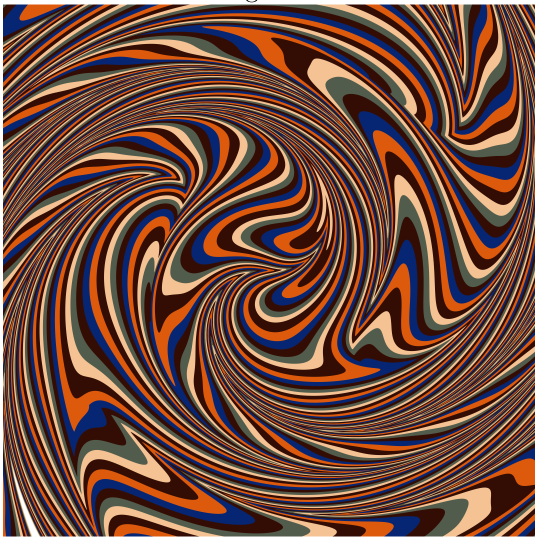
Figure 10

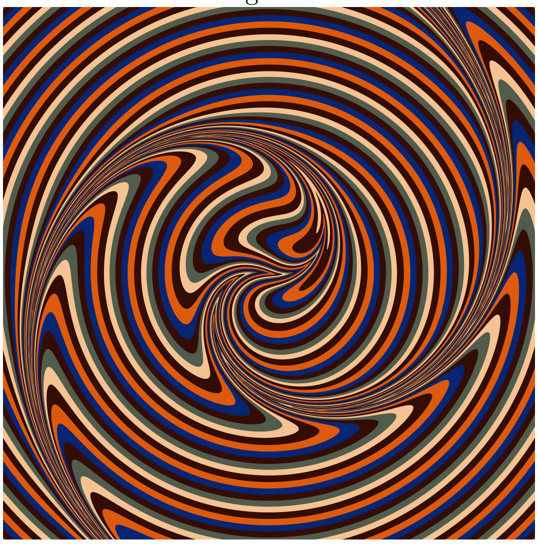
Figure 11


Figure 12

4. **French Curl**

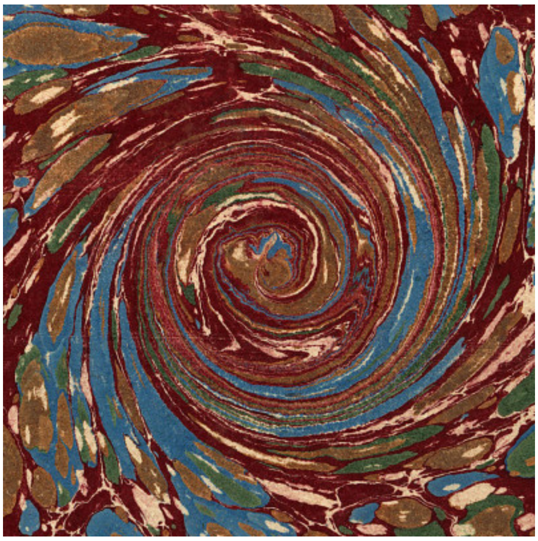

Figure 13

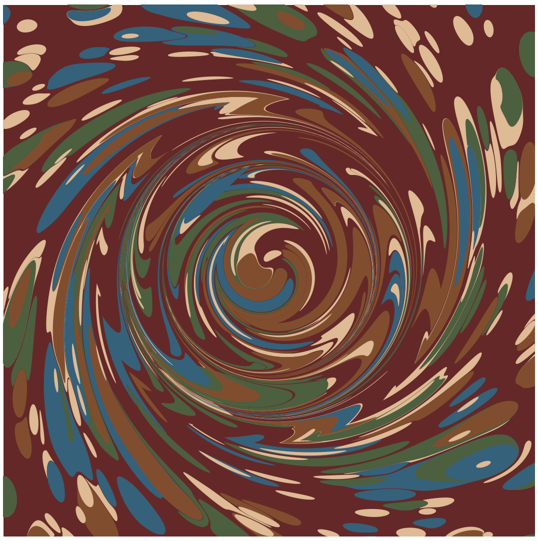

Figure 14

At first glance the photographic detail of an 1880 French Curl marbling[2] in Figure 13 looks as though it was produced by a vortex. The white nested chevrons show that it is instead the product of concentric circling by a stylus. Figure 14 is a mathematically produced marbling of a stylus circling four times at various radii; it has chevrons and much of the character of the original.

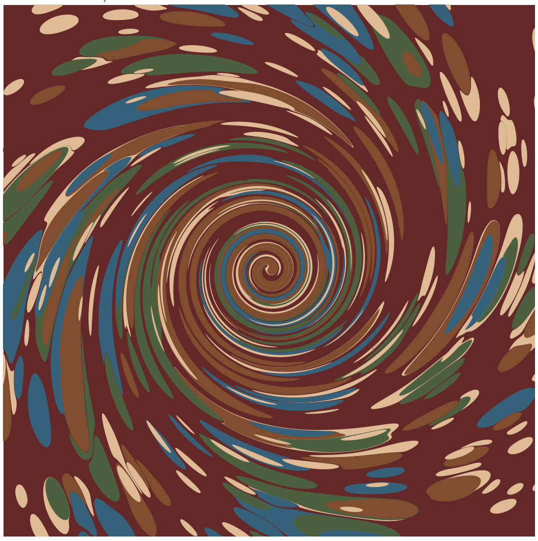

Figure 15

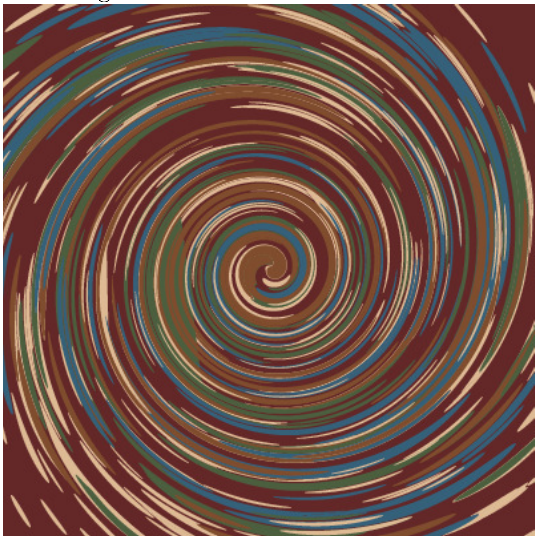

Figure 16

A Lamb-Oseen vortex captured immediately after the stirring at its center produces Figure 15, which bears some likeness to Figure 13. And this would be captured if the paper could be laid into and peeled off the tank quickly after stirring the vortex.

But within a minute, the rotational shear has propagated outward as shown in Figure 16, which no longer resembles Figure 13.

[2] Creator of the paper unknown. Scanned by Aristeas from a book in his own possession. [Public domain], from Wikimedia Commons

5. **Bubbles**

One way to produce vortexes in a Newtonian fluid is to move a stylus briskly in a straight line. If the Reynolds number exceeds 90, then it will shed Karman vortexes to alternating sides of the stylus path.

A 25 mm diameter dowel submerged 12.5 mm has a characteristic-length $D = 0.0042$m, the same as a half-submerged 25 mm diameter sphere. If the kinematic viscosity ($\nu = 0.001$ m^2/s) of the liquid is 1000 times that of water and the dowel is moved 20 cm/s through the liquid, then the Reynolds number is about 0.85, far less than the 90 needed to spawn vortexes. Re is inversely proportional to viscosity; reducing the viscosity by a factor of 10 raises Re to 8.5.

A half sphere of diameter 25 mm has buoyant-pressure (restoring force divided by cross-section area) of about 81 N/m^2. Surface tension pressure (restoring force divided by cross-section area) of water is roughly 3.7 N/m^2.

Drag is the force on the object moving through the tank fluid. There must be an equal and opposite net force on the liquid. Drag D for a sphere is the product of the friction coefficient C_D, frontal area ($\pi\, d^2/4$), and dynamic head $V^2\rho/2$ (for water $\rho = 997$ kg/m^3). That force divided by the frontal area of the object is a pressure (suction actually).

A bubble will be formed if this suction behind the moving stylus is larger than the sum of the restoring forces at the liquid surface.

For kinematic viscosity $\nu = 1000$ mm^2/s (1000 times that of water) the suction behind the 25 mm diameter dowel or sphere is 88 N/m^2, which exceeds the restoring pressures 81 N/m^2 and 3.7 N/m^2, and bubbles can result.

Kinematic viscosities below 50 mm^2/s (50 times that of water) would be needed to spawn vortexes in marbling. At 50 mm^2/s viscosity, a 25 mm cylinder would need to be moved at 20 cm/s over at least 16 cm before the first vortex was shed.

In water, a 5 mm cylinder moving at 2 cm/s would shed vortexes 3 cm apart. A 1 mm diameter stylus moved in a straight path at 5 cm/s would not shed vortexes.

So existing evidence of Karman (shed) vortexes is only likely to be found in marbling produced on a tank filled with water.

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