Modeling Thermal-Infrared Radiation from the Troposphere

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In the lower atmosphere ($Z = 16\,\mathrm{km}$), the moist adiabatic lapse rate L averages 6.5 K/km. So the temperature at altitude z is roughly $T_0 - L\,z$.

The air pressure P and density r at altitude z are

$$P(z) = P_0 \left(\frac{T_0 - L\,z}{T_0}\right)^{g/(LR)} \qquad r(z) = \frac{P_0}{R\cdot(T_0 - L\,z)}\left(\frac{T_0 - L\,z}{T_0}\right)^{g/(LR)} = \frac{P_0}{T_0 R}\left(\frac{T_0 - L\,z}{T_0}\right)^{g/(LR)-1}$$

where $R = 287\,\mathrm{m}^2{\cdot}\mathrm{s}^{-2}{\cdot}\mathrm{K}^{-1}$ is the gas constant for air, and g is the gravitational acceleration. $g/(LR) \approx 5.253$. Let $\rho(z, P_0, T_0) = r(z, T_0)/\nu$ be the density normalized so that Q, the integral of ρ over Z at $T_0 = 300$ K, is airmass of 1.

$$1 = Q(Z, P_0, T_0) = \int_0^Z \rho(z, P_0, T_0)\,dz = \int_0^Z \frac{P_0}{\nu T_0 R}\left(\frac{T_0 - L\,z}{T_0}\right)^{g/(LR)-1} dz = \frac{P_0}{\nu g}\cdot\left(1 - \left(\frac{T_0 - L\,Z}{T_0}\right)^{g/(LR)}\right)$$

$$\nu \approx 9.23 \times 10^3\,\mathrm{g}\cdot\mathrm{m}^{-4}$$

The saturation humidity decreases exponentially with temperature, hence it decreases exponentially with altitude. “Surface Dew Point and Water Vapor Aloft” [40] posits that vapor density (saturated or not) is exponentially decreasing through the troposphere.

Let W be the depth in millimeters of water from a vertical column of atmosphere were its water condensed. Let $V(z) = \upsilon\cdot(1 - e^{-\beta\,z})/\beta$ be the fraction of precipitable moisture depth from ground level to altitude z, where $\beta = 0.44\,\mathrm{km}^{-1}$ [40].

$$1 = \frac{\upsilon}{\beta}\,(1 - e^{-\beta\,Z}).$$

$$\text{For}\quad Z = 16\,\mathrm{km},\quad \upsilon = \frac{\beta}{(1 - e^{-\beta\,Z})} = 0.440\,\mathrm{km}^{-1}.$$

Let $\tau(z, \omega, \zeta, P_0, T_0, W)$ be the transmittance at wavenumber ω through an atmospheric column to altitude z. The logarithm of transmittance, K, is composed of dry and 1 mm of humidity components:

$$\begin{aligned}\tau(z, \omega, \zeta, P_0, T_0, W) &= e^{K(z,\omega,\zeta,P_0,T_0,W)}\\ K(z, \omega, \zeta, P_0, T_0, W) &= (\log B(\omega)\,Q(z, P_0, T_0) + \log H_1(\omega)\,W\,V(z))\cdot\alpha(\zeta)\end{aligned}$$

where $\alpha(\zeta)$ is the airmass at angle ζ from zenith. Two formulas for airmass are [29]:

$$\alpha(\zeta) = \frac{1}{\cos\zeta} \quad\text{and}\quad \alpha(\zeta) = \frac{1}{r}\left(\sqrt{\cos^2\zeta + 2r + r^2} - \cos\zeta\right) \quad\text{where}\quad r = \frac{8.75}{6378}$$

The choice does not materially effect the results of simulation.

The derivative of transmittance τ with respect to z is:

$$\begin{aligned}\kappa(z, \omega, \zeta, P_0, T_0, W) &= \frac{\partial\tau(z, \omega, \zeta, P_0, T_0, W)}{\partial z}\\ &= e^{K(z,\omega,\zeta,P_0,T_0,W)}\cdot K'(z, \omega, \zeta, P_0, T_0, W)\\ &= \tau(z, \omega, \zeta, P_0, T_0, W)\cdot K'(z, \omega, \zeta, P_0, T_0, W)\\ K'(z, \omega, \zeta, P_0, T_0, W) &= \frac{\partial K(z, \omega, \zeta, P_0, T_0, W)}{\partial z}\\ &= \left(\log B(\omega)\,\rho(z, P_0, T_0) + \log H_1(\omega)\,W\,\frac{\partial V(z)}{\partial z}\right)\cdot\alpha(\zeta)\end{aligned}$$

The attenuated emission per unit length at altitude z is $M(\omega, T_0 - L\,z) \cdot \kappa(z, \omega, \zeta, P_0, T_0, W)$. Thus the flow of thermal radiation from the cloudless troposphere into the emitter is:

$$S_Z(Z, \omega, \zeta, P_0, T_0, W) = \int_0^Z M(\omega, T_0 - L\,z) \cdot \kappa(z, \omega, \zeta, P_0, T_0, W)\, dz$$

The contributions from small z dominate the integral; so linear integration steps have poor numerical conditioning. $z = \exp y$ works, but then has too many steps near zero. Mike Speciner suggests hyperbolic sine as a compromise. Let $z = \gamma \cdot \sinh y$.

$$S_Z(Z, \omega, \zeta, P_0, T_0, W) = \int_0^{\sinh^{-1} \frac{Z}{\gamma}} M(\omega, T_0 - L \cdot \gamma \cdot \sinh y) \cdot \kappa(\gamma \cdot \sinh y, \omega, \zeta, P_0, T_0, W) \cdot \gamma \cdot \cosh y \cdot dy$$

Water and ice clouds act as blackbody radiators in the thermal-infrared band. For a cloud whose base is at altitude C:

$$\begin{aligned} S_C(C, \omega, \zeta, P_0, T_0, W) &= M(\omega, T_0 - L\,C) \cdot \tau(C, \omega, \zeta, P_0, T_0, W) \\ &+ \int_0^{\sinh^{-1} \frac{C}{\gamma}} M(\omega, T_0 - L\,\gamma \cdot \sinh y) \cdot \kappa(\gamma \cdot \sinh y, \omega, \zeta, P_0, T_0, W) \cdot \gamma \cdot \cosh y \cdot dy \end{aligned}$$

S_C and S_Z are the fluxes from a column of air at angle ζ from the zenith. In order to compute the total flux, integrate the product of the column flux with the emissivity $\varepsilon(\omega, \zeta)$ over the hemisphere and spectrum.

$$\begin{aligned} &\int_0^{\pi/2} \int_0^{\infty} 2\pi \cdot S(\omega, \zeta) \cdot \varepsilon(\omega, \zeta) \cdot \sin\zeta \cdot \cos\zeta \cdot \, d\omega\, d\zeta \\ &= \int_0^{\pi} \int_0^{\infty} \frac{\pi}{2} S(\omega, \zeta) \cdot \varepsilon(\omega, \zeta) \cdot \sin\zeta \cdot \, d\omega\, d\zeta \end{aligned}$$

The hourly thermal radiation from the troposphere is a mixture of the integrated S_C and S_Z according to the opaque-sky-cover ratio.

Net Radiative Transfer

$$\int_0^{\pi/2} \int_0^{\infty} 2\pi \cdot (M(\omega, T_0) - S(\omega, \zeta, T_0)) \cdot \varepsilon(\omega, \zeta) \cdot \sin\zeta \cdot \cos\zeta \cdot \, d\omega\, d\zeta$$

Integrating the net-radiative-transfer has the advantage that at ω where the troposphere is opaque, $M(\omega, T_0) - S(\omega, \zeta, T_0)$ is zero, allowing those iterations to be skipped.

The net-radiative-transfer for an emitter which is not at ambient temperature can be calculated by adding:

$$\int_0^{\pi/2} \int_0^{\infty} 2\pi \cdot (M(\omega, T_1) - M(\omega, T_0)) \cdot \varepsilon(\omega, \zeta) \cdot \sin\zeta \cdot \cos\zeta \cdot \, d\omega\, d\zeta$$

If ε is constant with respect to ω and ζ, this simplifies to: $(M_h(T_1) - M_h(T_0)) \cdot \varepsilon$, where M_h is the hemispheric black-body emission.

Restricted Aperture

For the case of a Lambertian emitter with non-spectral emissivity ε_L with an aperture restricted to a vertical θ-cone having apeture-gain G_A (between 0 and 1), the net radiative transfer is:

$$\int_0^{\theta/2} \int_0^{\infty} 2\pi \cdot (M(\omega, T_0) - S(\omega, \zeta)) \cdot \varepsilon_L \cdot G_A \cdot \sin\zeta \cdot \cos\zeta \cdot \, d\omega\, d\zeta$$

Bibliography

[29] M. Kenworthy. “Airmass due to the finite radius of the Earth”, 22nd Jan 2002, `http://mmtao.org/~mattk/docs/acc_airmass.pdf`

[40] Reitan, C., “Surface Dew Point and Water Vapor Aloft”, *Journal of Applied Meteorology*: Vol. 2, No. 6, pp. 776-779, 1963.