Modeling Thermal-Infrared Radiation from the Troposphere

Aubrey Jaffer
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In the lower atmosphere \((Z = 16\, \text{km})\), the moist adiabatic lapse rate \(L\) averages 6.5 K/km. So the temperature at altitude \(z\) is roughly \(T_0 - L\, z\).

The air pressure \(P\) and density \(\rho\) at altitude \(z\) are

\[
P(z) = P_0 \left(\frac{T_0 - L\, z}{T_0}\right)^{g/(LR)} \quad \rho(z) = \frac{P_0}{R \cdot (T_0 - L\, z)} \left(\frac{T_0 - L\, z}{T_0}\right)^{g/(LR) - 1}
\]

where \(R = 287 \, \text{m}^2\, \text{s}^{-2}\, \text{K}^{-1}\) is the gas constant for air, and \(g\) is the gravitational acceleration. \(g/(LR) \approx 5.253\).

Let \(\rho(z, P_0, T_0) = r(z, T_0)/\nu\) be the density normalized so that \(Q\), the integral of \(\rho\) over \(Z\) at \(T_0 = 300\, \text{K}\), is airmass of 1.

\[
1 = Q(Z, P_0, T_0) = \int_0^Z \rho(z, P_0, T_0) \, dz = \int_0^Z \frac{P_0}{\nu T_0 R} \left(\frac{T_0 - L\, z}{T_0}\right)^{g/(LR) - 1} \, dz = \frac{P_0}{\nu g} \left(1 - \left(\frac{T_0 - L\, z}{T_0}\right)^{g/(LR)}\right)
\]

\[
\nu \approx 9.23 \times 10^3 \, \text{g} \cdot \text{m}^{-4}
\]

The saturation humidity decreases exponentially with temperature, hence it decreases exponentially with altitude. “Surface Dew Point and Water Vapor Aloft” [40] posits that vapor density (saturated or not) is exponentially decreasing through the troposphere.

Let \(W\) be the depth in millimeters of water from a vertical column of atmosphere were its water condensed. Let \(V(z) = v \cdot (1 - e^{-\beta\, z})/\beta\) be the fraction of precipitable moisture depth from ground level to altitude \(z\), where \(\beta = 0.44\, \text{km}^{-1}\) [40].

\[
1 = \frac{v}{\beta} (1 - e^{-\beta\, z}).
\]

For \(Z = 16\, \text{km}\), \(v = \frac{\beta}{(1 - e^{-\beta\, Z})} = 0.440\, \text{km}^{-1}\).

Let \(\tau(z, \omega, \zeta, P_0, T_0, W)\) be the transmittance at wavenumber \(\omega\) through an atmospheric column to altitude \(z\). The logarithm of transmittance, \(K\), is composed of dry and 1 mm of humidity components:

\[
\tau(z, \omega, \zeta, P_0, T_0, W) = e^{K(z, \omega, \zeta, P_0, T_0, W)}
\]

\[
K(z, \omega, \zeta, P_0, T_0, W) = \left(\log B(\omega) \, Q(z, P_0, T_0) + \log H_1(\omega) \, W \, V(z)\right) \cdot \alpha(\zeta)
\]

where \(\alpha(\zeta)\) is the airmass at angle \(\zeta\) from zenith. Two formulas for airmass are [29]:

\[
\alpha(\zeta) = \frac{1}{\cos \zeta} \quad \text{and} \quad \alpha(\zeta) = \frac{1}{r} \left(\sqrt{\cos^2 \zeta + 2r + r^2} - \cos \zeta\right) \quad \text{where} \quad r = \frac{8.75}{6378}
\]

The choice does not materially effect the results of simulation.

The derivative of transmittance \(\tau\) with respect to \(z\) is:

\[
\kappa(z, \omega, \zeta, P_0, T_0, W) = \frac{\partial \tau(z, \omega, \zeta, P_0, T_0, W)}{\partial z} = e^{K(z, \omega, \zeta, P_0, T_0, W)} \cdot K'(z, \omega, \zeta, P_0, T_0, W)
\]

\[
K'(z, \omega, \zeta, P_0, T_0, W) = \frac{\partial K(z, \omega, \zeta, P_0, T_0, W)}{\partial z} = \left(\log B(\omega) \, \rho(z, P_0, T_0) + \log H_1(\omega) \, W \, \frac{\partial V(z)}{\partial z}\right) \cdot \alpha(\zeta)
\]
The attenuated emission per unit length at altitude \( z \) is \( M(\omega, T_0) - L z ) \cdot \kappa(z, \omega, \zeta, P_0, T_0, W) \). Thus the flow of thermal radiation from the cloudless troposphere into the emitter is:

\[
S_Z(Z, \omega, \zeta, P_0, T_0, W) = \int_0^Z M(\omega, T_0 - z) \cdot \kappa(z, \omega, \zeta, P_0, T_0, W) \, dz
\]

The contributions from small \( z \) dominate the integral; so linear integration steps have poor numerical conditioning. \( z = \exp y \) works, but then has too many steps near zero. Mike Speciner suggests hyperbolic sine as a compromise. Let \( z = \gamma \cdot \sinh y \).

\[
S_Z(Z, \omega, \zeta, P_0, T_0, W) = \int_0^{\sinh^{-1} \frac{Z}{\gamma}} M(\omega, T_0 - L \cdot \gamma \cdot \sinh y) \cdot \kappa(\gamma \cdot \sinh y, \omega, \zeta, P_0, T_0, W) \cdot \gamma \cdot \cosh y \cdot dy
\]

Water and ice clouds act as blackbody radiators in the thermal-infrared band. For a cloud whose base is at altitude \( C \):

\[
S_C(C, \omega, \zeta, P_0, T_0, W) = M(\omega, T_0 - L C) \cdot \tau(C, \omega, \zeta, P_0, T_0, W)
\]

\[
+ \int_0^{\sinh^{-1} \frac{C}{\gamma}} M(\omega, T_0 - L \cdot \gamma \cdot \sinh y) \cdot \kappa(\gamma \cdot \sinh y, \omega, \zeta, P_0, T_0, W) \cdot \gamma \cdot \cosh y \cdot dy
\]

\( S_C \) and \( S_Z \) are the fluxes from a column of air at angle \( \zeta \) from the zenith. In order to compute the total flux, integrate the product of the column flux with the emissivity \( \varepsilon(\omega, \zeta) \) over the hemisphere and spectrum.

\[
\int_0^{\pi/2} \int_0^\infty 2\pi \cdot S(\omega, \zeta) \cdot \varepsilon(\omega, \zeta) \cdot \sin \zeta \cdot \cos \zeta \cdot d\omega d\zeta
\]

\[
= \int_0^{\pi} \int_0^\infty \frac{\pi}{2} S(\omega, \zeta) \cdot \varepsilon(\omega, \zeta) \cdot \sin \zeta \cdot d\omega d\zeta
\]

The hourly thermal radiation from the troposphere is a mixture of the integrated \( S_C \) and \( S_Z \) according to the opaque-sky-cover ratio.

**Net Radiative Transfer**

\[
\int_0^{\pi/2} \int_0^\infty 2\pi \cdot (M(\omega, T_0) - S(\omega, \zeta, T_0)) \cdot \varepsilon(\omega, \zeta) \cdot \sin \zeta \cdot \cos \zeta \cdot d\omega d\zeta
\]

Integrating the net-radiative-transfer has the advantage that at \( \omega \) where the troposphere is opaque, \( M(\omega, T_0) - S(\omega, \zeta, T_0) \) is zero, allowing those iterations to be skipped.

The net-radiative-transfer for an emitter which is not at ambient temperature can be calculated by adding:

\[
\int_0^{\pi/2} \int_0^\infty 2\pi \cdot (M(\omega, T_1) - M(\omega, T_0)) \cdot \varepsilon(\omega, \zeta) \cdot \sin \zeta \cdot \cos \zeta \cdot d\omega d\zeta
\]

If \( \varepsilon \) is constant with respect to \( \omega \) and \( \zeta \), this simplifies to: \( (M_h(T_1) - M_h(T_0)) \cdot \varepsilon \), where \( M_h \) is the hemispheric black-body emission.

**Restricted Aperture**

For the case of a Lambertian emitter with non-spectral emissivity \( \varepsilon_L \) with an aperture restricted to a vertical \( \theta \)-cone having aperture-gain \( G_A \) (between 0 and 1), the net radiative transfer is:

\[
\int_0^{\theta/2} \int_0^\infty 2\pi \cdot (M(\omega, T_0) - S(\omega, \zeta)) \cdot \varepsilon_L \cdot G_A \cdot \sin \zeta \cdot \cos \zeta \cdot d\omega d\zeta
\]

**Bibliography**
