Canonical Rational Function Integration

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Abstract

The derivative of any rational function (ratio of polynomials) is a rational function. An algorithm and decision procedure for finding the rational function anti-derivative of a rational function is presented. This algorithm is then extended to derivatives of rational functions which include instances of a radical involving the integration variable. It is further extended to derivatives of rational functions which include instances of transcendental functions defined by separable first-order differential equations.

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1. Canonical form

In symbolic computation, a canonical form has only one representation for expressions which are equivalent. A less stringent constraint is normal form, where all expressions equivalent to 0 are represented by 0.

Richardson [1], Caviness [2], and Wang [3] explore which combinations of number systems, operations, algebraic, and transcendental functions are decidable (able to be resolved in finite time by an algorithm) for questions of an expression having a value greater than 0. Questions of ordering are valid only in ordered number systems such as the reals. Real analysis is an important domain, but not the only one.

Differential algebra (Ritt [4]) and Gröbner bases [5] are built from a polynomial ring whose polynomial coefficients are a field. The canonical form of the present work is built from the Euclidean domain (which is also a unique factorization domain and a commutative ring) of polynomials having integer coefficients.[1]

In the present work, each equation is in implicit form, a polynomial equated with zero. An expression is represented by a polynomial equation involving the special variable @. The ratio $a/(2+3\,b^2)$ is represented:

$$2\,@ + 3\,b^2\,@ - a = 0 \tag{1}$$

A square-root of a is represented by $@^2 - 2\,a = 0$. A root of the fifth-order polynomial $x^5 + x + 2\,a$ is represented by $@^5 + @ + 2\,a = 0$.

It is expected that the polynomial arithmetic algorithms will always combine the coefficients of the sum of terms with identical variable exponents, and remove monomial terms having 0 as coefficient. After a polynomial calculation is performed involving @, the polynomial is made square-free; that is, the exponent

[1] The coefficients being integer instead of rational is of practical importance to the speed of calculation. In order to keep the storage for rational numbers under control, polynomial arithmetic algorithms typically normalize the ratio of integers using greatest common divisor (GCD) after each arithmetic operation, trading time for storage. The storage required for the results of operations on integers grows more slowly than for rational numbers. In the present work, GCD operations on integers are postponed until the normalization phase.

of each factor is reduced to 1. Given polynomial equation $\mathtt{P}(@, x_1, \ldots)$, its square-free form is computed by (exactly) dividing $\mathtt{P}$ by the GCD of $\mathtt{P}$ and its derivative with respect to @:

$$\mathtt{P} \Big/ \gcd\left(\mathtt{P}, \frac{\mathrm{d}\,\mathtt{P}}{\mathrm{d}\,@}\right) \tag{2}$$

Each of the remaining factors of the square-free polynomial equation represents its degree of distinct solutions. The square-free form also removes factors not involving @, as well as the "content", the GCD of all the coefficients. Expressions equivalent to 0 will reduce to $@ = 0$; therefore, this representation is a normal-form.

The existence of total orderings for variables and terms is well established for Gröbner bases. Using a total ordering, the polynomial equation can be displayed in a deterministic way. The combination of a total ordering and this implicit normal form is thus a canonical form for expressions.

The canonical form can be extended to equations not involving @. In this case, the canonicalization procedure makes all factors involving variables square-free. The variable ordering governs which high order term has a positive coefficient.

2. **Rational function differentiation**

Let

$$f(x) = \prod_{j \neq 0} p_j(x)^j \tag{3}$$

be a rational function of x where the primitive polynomials $p_j(x)$ are square-free and mutually relatively prime.

The derivative of $f(x)$ is

$$\frac{\partial f}{\partial x}(x) = \sum_j j\, p_j(x)^{j-1}\, p_j'(x) \prod_{k \neq j} p_k(x)^k \tag{4}$$

Lemma 1. *Given square-free and relatively prime primitive polynomials $p_j(x)$, $\sum_j j\, p_j'(x) \prod_{k \neq j} p_k(x)$ has no factors in common with $p_j(x)$.*

Assume that the sum has a common factor $p_h(x)$ such that $p_h(x)$ divides the sum:

$$p_h(x) \,\Bigg|\, \sum_j j\, p_j'(x) \prod_{k \neq j} p_k(x)$$

$p_h(x)$ divides all terms for $j \neq h$. Because it divides the whole sum, $p_h(x)$ must divide the remaining term $h\, p_h'(x) \prod_{k \neq h} p_k(x)$. From the given conditions, $p_h(x)$ does not divide $p_h'(x)$ because $p_h(x)$ is square-free; and $p_h(x)$ does not divide $p_k(x)$ for $k \neq h$ because they are relatively prime.

3. Rational function integration

Separating square-and-higher factors from the sum in equation (4):

$$\frac{\partial f}{\partial x}(x) = \left[\prod_j p_j(x)^{j-1}\right] \left[\sum_j j\, p_j'(x) \prod_{k\neq j} p_k(x)\right] \tag{5}$$

There are no common factors between the sum and product terms of equation (5) because of the relatively prime condition of equation (3) and because of Lemma 1. Hence, this equation cannot be reduced and is canonical.

Split equation (5) into factors by the sign of the exponents, giving:

$$\frac{\partial f}{\partial x}(x) = \frac{\prod_{j>2} p_j(x)^{j-1}}{\prod_{j<0} p_j(x)^{1-j}} \overbrace{p_2(x) \sum_j j\, p_j'(x) \prod_{k\neq j} p_k(x)}^{\mathbf{L}} \tag{6}$$

The denominator is $\prod_{j\leq 0} p_j(x)^{1-j}$. Its individual $p_j(x)$ can be separated by square-free factorization. The $p_j(x)$ for $j > 2$ can also be separated by square-free factorization of the numerator. Neither $p_2(x)$ nor $\sum_j j\, p_j'(x) \prod_{k\neq j} p_k(x)$ have square factors; so square-free factorization will not separate them. Treating $p_2(x)$ as 1 lets its factor be absorbed into $p_1(x)$. Note that $p_j(x) = 1$ for factor exponents j which don't occur in the factorization of $\partial f/\partial x$. All the $p_j(x)$ are now known except $p_1(x)$. Once $p_1(x)$ is known, $f(x)$ can be recovered by equation (3). Let polynomial `L` be the result of dividing the numerator of $\partial f/\partial x$ by $\prod_{j>2} p_j(x)^{j-1}$.

$$\overbrace{\sum_j j\, p_j'(x) \prod_{k\neq j} p_k(x)}^{\mathbf{L}} = \overbrace{\sum_{j\neq 1} j\, p_j'(x) \prod_{1\neq k\neq j} p_k(x)}^{\mathbf{M}}\; p_1(x) + p_1'(x) \overbrace{\prod_{k\neq 1} p_k(x)}^{\mathbf{N}} \tag{7}$$

Because they don't involve $p_1(x)$, polynomials `M` and `N` in equation (7) can be computed from the square-free factorizations of the numerator and denominator. This allows $p_1(x)$ to be constructed by a process resembling long division. The trick at each step is to construct a monomial $q(x)$ such that $\mathtt{M}\, q(x) + q'(x)\, \mathtt{N}$ cancels the highest term of dividend `R` (which is initially `L`).

Let $\deg(p)$ be the degree of x in polynomial p. Let $\mathrm{coeff}\,(p, w)$ be the coefficient of the x^w term of polynomial p for non-negative integer w.

Note that $\deg(\mathtt{M}) = \deg(\mathtt{N}) - 1$ because the derivative of exactly one of the $p_j(x)$ occurs instead of $p_j(x)$ in each term of `M`. And $\deg(q(x)\,\mathtt{M}) = \deg(q'(x)\,\mathtt{N})$ because $\deg(q'(x)) = \deg(q(x)) - 1$.

The polynomial $p_1(x)$ can be constructed by the following procedure. Let `A`, `C`, and `R` be rational expressions. Only the numerators of `A` and `R` contain powers of x. Starting from polynomials `L`, `M`, and `N`:

```
A = 0
R = L
Nxd = deg(N)
while ( g = deg(num(R)) - Nxd + 1 ) >= 0:
    Rxd = deg(num(R))
    RxC = coeff(num(R),Rxd)
    C = RxC / ( coeff(M,Nxd-1) + g*coeff(N,Nxd) ) / denom(R)
    A = A + C * x^g
    R = R - C * ( M*x^g + N*diff(x^g,x) )
    if deg(num(R)) > Rxd:
        fail
    if 0 == R:
        return A
```

At the end of this process, if $\mathtt{R} = 0$, then $p_1(x)$ is the numerator of `A`; and the anti-derivative is $f(x) = \prod_j p_j(x)^j / \mathrm{denom}\,(\mathtt{A})$. Otherwise the anti-derivative is not a rational function.

Just as this algorithm works with $p_2(x)$ absorbed into $p_1(x)$, it works with all of the $p_j(x)$ for $j > 1$ absorbed into $p_1(x)$. This removes the need to factor the numerator and provides the opportunity to enhance the algorithm to handle algebraic extensions.

4. Algebraic extension

Let variable y represent one of the solutions of its defining equation (reduction rule) represented by a polynomial $\mathtt{Y} = 0$ which is irreducible over the integers. For example $\mathtt{Y}$ would be $y^3 - x$ for a cube root of x.

In order to normalize polynomials with regard to Y, each polynomial P containing y is replaced by prem(P, Y), the remainder of pseudo-division of P by Y, as described by Knuth [6].

While that process normalizes polynomials, it doesn't normalize ratios of polynomials, for instance:

$$1/y^2 = 1/(\sqrt[3]{x})^2 = \sqrt[3]{x}/x = y/x$$

After the polynomials are normalized, if the denominator still contains the extension y, it is possible to move y to the numerator by multiplying both numerator and denominator by the y-conjugate of the denominator, then normalizing both numerator and denominator by Y. The conjugate of a polynomial P with respect to Y can be computed by the following procedure where deg(Q) is the degree of y in polynomial Q and pquo(Y, P) and prem(Y, P) are the quotient and remainder of pseudo-division of Y by P:

```
def conj(P):
    if deg(P) < deg(Y):
        Q = pquo(Y,P)
        R = prem(Y,P)
    else:
        Q = 1
        R = 0
    if deg(R) == 0:
        return Q
    else:
        return Q * conj(R)
```

As discussed by Caviness and Fateman [7], multiple ring extensions involving the same variable can be combined into a single extension. For the purposes of integration, combine the ring extensions involving the variable of integration x into a single variable y with its defining equation Y.

5. Rational function integration with algebraic extension

With a single algebraic extension y which is a function of x, and the denominator free of y, and all the numerator factors in $p_1(x,y)$, the previous development can be reformulated:

$$f(x,y) = \prod_{j\le 1} p_j(x,y)^j \tag{8}$$

The derivative of $f(x,y)$ with respect to x is

$$\frac{\partial f}{\partial x}(x,y) = \sum_{j\le 1} j\, p_j(x,y)^{j-1}\, p_j'(x,y) \prod_{k\ne j} p_k(x,y)^k \tag{9}$$

Separating into numerator and denominator:

$$\frac{\partial f}{\partial x}(x,y) = \frac{\sum_j j\, p_j'(x,y) \prod_{k\ne j} p_k(x,y)}{\prod_{j\le 0} p_j(x,y)^{1-j}} \tag{10}$$

This time, $\mathtt{L}$ is the whole numerator of equation (10). Note that the denominator includes $p_0(x,y)$; $p_0(x,y)$ does not contribute a term to $\mathtt{M}$ because its coefficient j is 0. Separating $p_1(x,y)$ from the denominator factors:

$$\overbrace{\sum_j j\, p_j'(x,y) \prod_{k\ne j} p_k(x,y)}^{\mathbf{L}} = \overbrace{\sum_{j\le 0} j\, p_j'(x,y) \prod_{k\ne j} p_k(x,y)}^{\mathbf{M}}\; p_1(x,y) + p_1'(x,y) \overbrace{\prod_{k\le 0} p_k(x,y)}^{\mathbf{N}} \tag{11}$$

Because they don't involve $p_1(x,y)$, polynomials $\mathtt{M}$ and $\mathtt{N}$ can be computed from the square-free factorization of the denominator. The trick at each step is to construct a polynomial t such that $\mathtt{M}\,t + t'\,\mathtt{N}$ cancels the highest term of dividend $\mathtt{R}$ (initial $\mathtt{R} = \mathtt{L}$).

For polynomial q, $\deg(q'(x)) = \deg(q(x)) - 1$ in the previous section. The derivatives of polynomials involving x and its algebraic extension y are more complicated. The derivative of y is found by differentiating the y defining equation $\mathtt{Y} = 0$, then eliminating y' from the chain rule for each term of the polynomial it occurs in. An example using the square of $y = (x^4 + a)^{1/3}$ demonstrates the reduction:

$$\frac{\mathrm{d}\, y^2}{\mathrm{d}\, x} = \frac{8\, x^3}{3\,(x^4+a)^{1/3}} = \frac{8\, x^3\,(x^4+a)^{2/3}}{3\,(x^4+a)} = \frac{8\, x^3\, y^2}{3\,(x^4+a)}$$

The degree of y does not change between y^2 and $\mathrm{d}\, y^2/\,\mathrm{d}\, x$. However the degree of x decreases; the degree of the denominator is one more than the degree of the numerator of $\mathrm{d}\, y^2/\,\mathrm{d}\, x$. This holds for any algebraic extension defined by a primitive polynomial.

Let $\mathtt{A}$, $\mathtt{C}$, and $\mathtt{R}$ be rational expressions. Let $\mathtt{Q}$ and $\mathtt{T}$ be polynomials of x containing no algebraic extensions of x. Let $g = \deg_x \mathtt{R} - \deg_x \mathtt{N} + 1$. The addition of 1 is to compensate for the reduction in the degree of x in $p_1'(x,y)$.

When there is no algebraic extension, $t = x^g$. If there is an algebraic extension y, let q be the denominator of normalized $\mathrm{d}\, y/\,\mathrm{d}\, x$, i be the integer quotient $g/\deg_x q$, and set g to the remainder of $g/\deg_x q$. Then:

$$t = q^i\, x^g\, y^h$$

The polynomial $p_1(x,y)$ can be constructed by the following procedure given polynomials $\mathtt{L}$, $\mathtt{M}$, and $\mathtt{N}$:

```
A = 0
R = L
Q = denom( normalize( diff(y,x) ) )
Nyd = deg(N,y)
NyC = coeff(N,y,Nyd)
Nxd = deg(NyC,x)
```

```
while 0 < 1:
    Ryd = deg(num(R),y)
    RyC = coeff(num(R),y,Ryd)
    Rxd = deg(RyC,x)
    h = Ryd - Nyd
    g = ( Rxd - Nxd + 1 )
    if 0 = deg(Q,x):
        T = x^g
    else:
        i = quotient(g,deg(Q,x))
        g = remainder(g,deg(Q,x))
        T = Q^i * x^g * y^h
    B = normalize( N*diff(T,x) + M*T )
    C = coeff(RyC,x,Rxd) * denom(B) / denom(R) /
        coeff(coeff(num(B),y,Ryd),x,Rxd)
    A = A + C * T
    R = R - C * B
    if 0 = R:
        return A
    if deg(num(R),y) > Ryd:
        fail
    if deg(num(R),y) == Ryd and
       deg(coeff(num(R),y,deg(num(R),y)),x) >= Rxd:
        fail
```

The looping continues only as long as the degree of `R` decreases. If this process succeeds, then the numerator of `A` is $p_1(x,y)$; and the anti-derivative is $f(x,y) = \mathtt{A} \prod_{j<1} p_j(x,y)^j$.

6. Transcendental extension

Some transcendental extensions involving a variable can be described by separable first-order differential equations:

$$\log(x)' = \frac{1}{x}\,x' \tag{12}$$

Where a is independent of x, applying the chain-rule:

$$\log(x^a)' = \frac{a\,x^{a-1}}{x^a}\,x' = \frac{a}{x}\,x' \tag{13}$$

Eliminating x'/x from equations (12) and (13) results in:

$$\log(x^a)' = a\log(x)'$$

Taking the anti-derivative of both sides:

$$\log(x^a) = a\,\log(x) + C \tag{14}$$

Thus, $\log(x)$ is sufficient to be the canonical extension for both $\log(x)$ and $\log(x^a)$. Logarithm is multi-valued in the complex plane. Like radical functions, $\log(x)$ assumes one branch. The canonical form described here produces equations which are true for any branch of log assuming all references to $\log(x)$ are the same branch. Note that the branch $\log(x)$ is not linked to the branch of $\log(y)$.

To describe the inverse function of $\log(x)$, substitute y for $\log(x)$ and $\exp(y)$ for x:

$$\frac{y'}{\exp(y)'} = \frac{1}{\exp(y)} \qquad \frac{\exp(y)'}{y'} = \exp(y) \tag{15}$$

Part of normalization for an expression (or equation) reduces the expression modulo the defining rules of the extensions appearing in that expression. For a transcendental extension, this is its defining differential equation.

These reductions serve to normalize expressions (involving variables) with unnested extensions. While (irreducible) algebraic expressions involving unnested transcendental extensions are canonical, when a transcendental function is composed with its inverse, an algebraic expression (without transcendental extensions) can result.

7. Reduction of nested transcendental extensions

For $\log(\exp(y))$, the reduction results from integrating the combination of defining differential equations (12) and (15) with $x = \exp(y)$:

$$\log(\exp(y))' = \frac{\exp(y)'}{\exp(y)} = y' \qquad \log(\exp(y)) = y + C$$

For $\exp(\log(x))$, the defining differential equations (12) and (15) combined with $y = \log(x)$ is separated, integrated ($\ln[= \log]$ is the result of the integration), then exponentiated:

$$\frac{\exp(\log(x))'}{\exp(\log(x))} = \frac{x'}{x} \qquad \ln(\exp(\log(x))) = \ln(x) \qquad \exp(\log(x)) = x$$

The defining equations for arc-tangent and tangent are:

$$\arctan(x)' = \frac{x'}{x^2+1} \qquad \frac{\tan(\theta)'}{\theta'} = \tan(\theta)^2 + 1 \tag{16}$$

$$x = \tan(\theta) \qquad \arctan(\tan(\theta))' = \frac{\tan(\theta)'}{\tan(\theta)^2+1} = \theta' \qquad \arctan(\tan(\theta)) = \theta + C$$

$$\theta = \arctan(x) \qquad \frac{\tan(\arctan(x))'}{\arctan(x)'} = \tan(\arctan(x))^2 + 1 \qquad \frac{\tan(\arctan(x))'}{\tan(\arctan(x))^2+1} = \frac{x'}{x^2+1}$$

For both $\tan(\arctan(x))$ and $\exp(\log(x))$, composition through the defining equations results in a form with polynomial function Φ:

$$\frac{y'}{\Phi(y)} = \frac{x'}{\Phi(x)} \tag{17}$$

Equation (17) is a stronger constraint than $y' = x'$ in that it implies $y = x$ without a constant of integration. Equation (17) is not the only form reducing to an algebraic relationship between y and x. Equation (18) where b is a nonzero integer may also reduce to an algebraic relation:

$$\frac{y'}{\Phi(y)} = \frac{b\,x'}{\Phi(x)} \tag{18}$$

Separating $y(x)$ into a ratio of relatively prime polynomials $y(x) = f(x)/g(x)$:

$$\frac{(f/g)'}{\Phi(f/g)} = \frac{f'\,g - f\,g'}{g^2\,\Phi(f/g)} = \frac{b\,x'}{\Phi(x)} \qquad (f'\,g - f\,g')\,\Phi(x) = b\,g^2\,\Phi(f/g)\,x' \tag{19}$$

f, f', g, g', x, x', and $\Phi(x)$ are all polynomials. Thus, $(f'\,g - f\,g')\,\Phi(x)$ is a polynomial. If equation (19) is to hold, then $g^2\,\Phi(f/g)\,x'$ must also be a polynomial, which will only be the case when the denominator of $\Phi(f/g)$ has degree 1 or 2 times the degree of g. When $g^2\,\Phi(f/g)\,x'$ does not equal $(f'\,g - f\,g')\,\Phi(x)$, the scaled composition cannot be reduced to an algebraic relation.

For both $\Phi(y) = y$ and $\Phi(y) = y^2 \pm 1$, it is the case that $g^2\,\Phi(f_b/g_b) = \Phi(x)^b$, and Φ is related to $y_b = f_b/g_b$ with $b \geq 2$:

$$(f_b'\,g_b - f_b\,g_b')\;\Phi(x) = b\,\Phi(x)^b\,x' \quad \rightarrow \quad f_b'\,g_b - f_b\,g_b' = b\,\Phi(x)^{b-1}\,x'$$

In the case of $\Phi(y) = y$,

$$y_1 = x \qquad y_b = y_{b-1}\,x \qquad \frac{y'}{y} = \frac{b\,x'}{x} \to y = x^b \tag{20}$$

In the case of $\Phi(y) = 1 + y^2$, using the sum-of-angles formula (a la Chebyshev):

$$\tan(b\,\theta) = \frac{\tan((b-1)\,\theta) + \tan(\theta)}{1 - \tan((b-1)\,\theta)\ \tan(\theta)} \tag{21}$$

Let $\theta = \arctan x$, $y_1 = x$, $y_b = f_b/g_b$:

$$y_b = \frac{y_{b-1} + x}{1 - y_{b-1}\,x} = \frac{f_{b-1} + g_{b-1}\,x}{g_{b-1} - f_{b-1}\,x} \qquad y_{-b} = \frac{y_{-b+1} - x}{1 + y_{-b+1}\,x} = \frac{g_{-b+1}\,x - f_{-b+1}}{g_{-b+1} + f_{-b+1}\,x} \tag{22}$$

Because $f_1 = x$ and $g_1 = 1$, the difference of the degrees of f_b and g_b alternate between 1 and -1 with the parity of b.

Extensions for irreducible Φ polynomials will always reduce to identity when directly composed with their inverse function; those which can reduce scaled composition to more complicated algebraic expressions are limited to degrees of 1 or 2 by equation (19). Any y_b recurrence must be symmetrical in x and y_{b-1}. There are few symmetrical candidates for y_b which might have transcendental-to-algebraic reductions. They were checked by formula (19) for $y_2(x) = f_2/g_2$.

For hyperbolic tangent $y_b = \tanh(b\ \mathrm{atanh}\,x)$, $\Phi(y) = 1 - y^2$, $y_1 = x$, $y = f/g$ and:

$$y_b = \frac{y_{b-1} + x}{1 + y_{b-1}\,x} = \frac{f_{b-1} + g_{b-1}\,x}{g_{b-1} + f_{b-1}\,x} \qquad y_{-b} = \frac{y_{-b+1} - x}{1 - y_{-b+1}\,x} = \frac{g_{-b+1}\,x - f_{-b+1}}{g_{-b+1} - f_{-b+1}\,x} \tag{23}$$

But $\Phi(x) = 1 - x^2$ is reducible, having factors $1 - x$ and $1 + x$; so it is not associated with a lone canonical extension. Instead the integral uses extensions $\log(x+1)$ and $\log(x-1)$:

$$\int \frac{\mathrm{d}\,x}{1 - x^2} = \frac{\log(x+1) - \log(x-1)}{2}$$

Equation (17) is symmetrical. Both sides can be scaled by nonzero integers:

$$\frac{a\,y'}{\Phi(y)} = \frac{b\,x'}{\Phi(x)} \tag{24}$$

A composition of defining differential equations yielding a form (24) results in a relation $y_b = x_a$. For $\Phi(t) = t$, $y = (\sqrt[a]{x})^b$. For other Φ, the result is a polynomial relation $y_b = x_a$.

Because $x'/\Phi(x) + x'/\Phi(x) = 2\,x'/\Phi(x)$, using equations (20) and (22) without recurrence directs how to compose with a sum of inverse transcendental functions.

$$\begin{gathered}\exp(\log x_1 + \log x_2) \to y = x_1\,x_2 \\ \tan(\arctan x_1 + \arctan x_2) \to y = (x_1 + x_2)/(1 - x_1\,x_2)\end{gathered} \tag{25}$$

A related problem is to normalize $\log(x\,z) \to \log x + \log z$. Taking the total differential of $\log(x\,z)$ yields:

$$\log(x\,z)' = \frac{x'\,z + x\,z'}{x\,z} = \frac{x'}{x} + \frac{z'}{z}$$

which is separable and integrable to $\log x + \log z$. This procedure works for arctan() as well:

$$\int \arctan\left(\frac{x+z}{1 - x\,z}\right)' = \int \frac{x'}{1+x^2} + \int \frac{z'}{1+z^2} = \arctan x + \arctan z$$

These example nested transcendental functions simplify to rational functions (polynomial ratios):

$$\begin{aligned}\exp(3\ \log x) &\to \frac{y'}{y} = \frac{3\,x'}{x} \to y_3 = x^3 \\ \tan(4\ \arctan x) &\to \frac{y'}{1+y^2} = \frac{4\,x'}{1+x^2} \to y_4 = \frac{4\,x - 4\,x^3}{1 - 6\,x^2 + x^4} \\ \tan(7\ \arctan x) &\to \frac{y'}{1+y^2} = \frac{7\,x'}{1+x^2} \to y_7 = \frac{-7\,x + 35\,x^3 - 21\,x^5 + x^7}{-1 + 21\,x^2 - 35\,x^4 + 7\,x^6}\end{aligned}$$

8. Nested transcendental extensions

The Lambert W function is an inverse of $y \exp y$. Its derivative is used as its defining differential equation:

$$\frac{\partial\,\mathrm{W}(z)}{\partial z} = \frac{\mathrm{W}(z)}{z\,[1+\mathrm{W}(z)]} \tag{26}$$

$$\frac{w'}{z'} = \frac{w}{z\,[1+w]} \qquad \left[1+\frac{1}{w}\right]\, w' = \frac{z'}{z} \qquad w + \ln w = \ln z \qquad z = w\,\exp w$$

The derivative of the composition of W with its inverse, $x\,\exp x$, reduces to a separable differential equation:

$$\frac{\mathrm{W}(x\,\exp x)+1}{\mathrm{W}(x\,\exp x)}\,\mathrm{W}(x\,\exp x)' = \frac{x+1}{x}\,x' \tag{27}$$

Equation (27) is of the form:

$$\frac{\mathrm{W}'}{\Phi(\mathrm{W})} = \frac{x'}{\Phi(x)} \qquad \Phi(y) = \frac{y}{y+1} \tag{28}$$

Thus, $\mathrm{W}(x\,\exp x) = x$. Because $\Phi(y)$ is not a polynomial, reductions from forms like equation (24) are not possible for W. Differentiating $\mathrm{W}(x)\,\exp(\mathrm{W}(x))$, then reducing by equations (15) and (26):

$$\begin{aligned}(\mathrm{W}(y)\;\exp(\mathrm{W}(y)))' &= \exp(\mathrm{W}(y))\;\,\mathrm{W}(y)' + \exp(\mathrm{W}(y))'\;\,\mathrm{W}(y)\\ &= \frac{\exp(\mathrm{W}(y))\,[1+\mathrm{W}(y)]\;\,\mathrm{W}(y)\,y'}{y\,[1+\mathrm{W}(y)]}\end{aligned} \tag{29}$$

$$\frac{(\mathrm{W}(y)\;\exp(\mathrm{W}(y)))'}{\mathrm{W}(y)\;\exp(\mathrm{W}(y))} = \frac{y'}{y} \tag{30}$$

Thus, $\mathrm{W}(x)\,\exp(\mathrm{W}(x)) = x$.

9. Transcendental constants

This system is canonical for algebraic extensions involving variables. It cannot be canonical for algebraic constants because the many roots of unity make alternate factorizations possible, and the system requires that polynomials be a unique factorization domain. Algebraic constants can be represented; but not all possible reductions will happen as part of the normalization processes.

This is also the reason that trigonometric functions were not encoded with log() and exp() using imaginary transcendental constants ($\pi\,\mathrm{i}$).

Transcendental constants can be represented by an uninstantiated differential equation and a rational argument value:

$$\exp 1 = \mathrm{e} \qquad \arctan 1 = \pi/4$$

If a transcendental function is composed with the transcendental constant, then their differential equations are combined. If the resulting equation is separable, then the recurrence is used to construct the algebraic result as before. $\exp 0 = 1$ and $\arctan 0 = 0$ are deducible from the recurrence without explicit training.

10. Rational function integration with transcendental extension

For an integer power w of an algebraic extension $y(x)$:

$$\deg_y \frac{\partial y^w}{\partial x} = w \qquad \deg_x \frac{\partial y^w}{\partial x} = -1$$

Transcendental extensions behave differently:

$y = \exp(x^w)$	$\deg_y \frac{\partial y}{\partial x} = 1$	$\deg_x \frac{\partial y}{\partial x} = w-1$
$y = \log^w(x)$	$\deg_y \frac{\partial y}{\partial x} = w-1$	$\deg_x \frac{\partial y}{\partial x} = -1$

Table 1 transcendental extensions

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