Canonical Rational Function Integration
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Abstract
The derivative of any rational function is a rational function. An algorithm and decision procedure for finding the rational function anti-derivative of a rational function is presented. This algorithm is then extended to derivatives of rational functions including instances of a radical involving the integration variable.

1. Rational function differentiation
Let
\[ f(x) = \prod_{j \neq 0} p_j(x)^j \] (1)
be a rational function of \( x \) where the primitive polynomials \( p_j(x) \) are square-free and mutually relatively prime.

The derivative of \( f(x) \) is
\[ \frac{df}{dx}(x) = \sum_j j p_j(x)^{j-1} p'_j(x) \prod_{k \neq j} p_k(x)^k \] (2)

Lemma 1. Given square-free and relatively prime primitive polynomials \( p_j(x) \), \( \sum_j j p'_j(x) \prod_{k \neq j} p_k(x) \) has no factors in common with \( p_j(x) \).

Assume that the sum has a common factor \( p_h(x) \) such that:
\[ p_h(x) \mid \sum_j j p'_j(x) \prod_{k \neq j} p_k(x) \]

\( p_h(x) \) divides all terms for \( j \neq h \). Because it divides the whole sum, \( p_h(x) \) must divide the remaining term \( h p'_h(x) \prod_{k \neq h} p_k(x) \). From the given conditions, \( p_h(x) \) does not divide \( p'_h(x) \) because \( p_h(x) \) is square-free; and \( p_h(x) \) does not divide \( p_k(x) \) for \( k \neq h \) because they are relatively prime.
2. Rational function integration

Separating square-and-higher factors from the sum in equation (2):

\[ \frac{df}{dx}(x) = \left[ \prod_j p_j(x)^{j-1} \right] \left[ \sum_j j \left( \prod_{k \neq j} p_k(x) \right) \right] \]  

(3)

There are no common factors between the sum and product terms of equation (3) because of the relatively prime condition of equation (1) and because of Lemma 1. Hence, this equation cannot be reduced and is canonical.

Split equation (3) into factors by the sign of the exponents, giving:

\[ \frac{df}{dx}(x) = \prod_{j>0} p_j(x)^{j-1} p_2(x) \sum_j j p_j'(x) \prod_{k \neq j} p_k(x) \]  

(4)

The denominator is \( \prod_{j<0} p_j(x)^{1-j} \). Its individual \( p_j(x) \) can be separated by square-free factorization. The \( p_j(x) \) for \( j > 2 \) can also be separated by square-free factorization of the numerator. Neither \( p_2(x) \) nor \( \sum j p_j'(x) \prod_{k \neq j} p_k(x) \) have square factors; so square-free factorization will not separate them. Treating \( p_2(x) \) as 1 lets its factor be absorbed into \( p_1(x) \). Note that \( p_j(x) = 1 \) for factor exponents \( j \) which don’t occur in the factorization of \( df/dx \). All the \( p_j(x) \) are now known except \( p_1(x) \). Once \( p_1(x) \) is known, \( f(x) \) can be recovered by equation (1). Let polynomial \( L \) be the result of dividing the numerator of \( df/dx \) by \( \prod_{j>2} p_j(x)^{j-1} \).

\[ \sum_j j p_j'(x) \prod_{k \neq j} p_k(x) = \sum_j j p_j'(x) \prod_{k \neq j} p_k(x) p_1(x) + p_1'(x) \prod_{k \neq j} p_k(x) \]  

(5)

Because they don’t involve \( p_1(x) \), polynomials \( M \) and \( N \) in equation (5) can be computed from the square-free factorizations of the numerator and denominator. This allows \( p_1(x) \) to be constructed by a process resembling long division. The trick at each step is to construct a monomial \( q(x) \) such that \( M q(x) + q'(x) N \) cancels the highest term of dividend \( R \) (which is initially \( L \)).

Let \( \deg(p) \) be the degree of \( x \) in polynomial \( p \). Let \( \text{coeff}(p, w) \) be the coefficient of the \( x^w \) term of polynomial \( p \) for non-negative integer \( w \).

Note that \( \deg(M) = \deg(N) - 1 \) because the derivative of exactly one of the \( p_j(x) \) occurs instead of \( p_j(x) \) in each term of \( M \). And \( \deg(q(x) M) = \deg(q'(x) N) \) because \( \deg(q'(x)) = \deg(q(x)) - 1 \).

The polynomial \( p_1(x) \) can be constructed by the following procedure. Let \( A \), \( C \), and \( R \) be rational expressions. Only the numerators of \( A \) and \( R \) contain powers of \( x \). Starting from polynomials \( L \), \( M \), and \( N \):

\[
\begin{align*}
A &= 0 \\
R &= L \\
N x d &= \deg(N) \\
\text{while ( } g = \deg(\text{num}(R)) - N x d + 1 \text{ ) } >= 0: \\
R x d &= \deg(\text{num}(R)) \\
R x C &= \text{coeff}(\text{num}(R), R x d) \\
C &= R x C / ( \text{coeff}(N, N x d - 1) + g \ast \text{coeff}(N, N x d) ) / \text{denom}(R) \\
A &= A + C \ast x^g \\
R &= R - C \ast ( M \ast x^g + N \ast \text{diff}(x^g, x) ) \\
\text{if deg(\text{num}(R)) } > R x d: \\
\text{fail} \\
\text{if 0 == R:} \\
\text{return } A \\
\end{align*}
\]

At the end of this process, if \( R = 0 \), then \( p_1(x) \) is the numerator of \( A \); and the anti-derivative is \( f(x) = \prod_j p_j(x)^j / \text{denom}(A) \). Otherwise the anti-derivative is not a rational function.

Just as this algorithm works with \( p_2(x) \) absorbed into \( p_1(x) \), it works with all of the \( p_j(x) \) for \( j > 1 \) absorbed into \( p_1(x) \). This removes the need to factor the numerator and provides the opportunity to enhance the algorithm to handle algebraic field extensions.
3. **Algebraic field extension**

Let \( y \) be a variable representing one of the solutions of its defining equation (reduction rule) represented by a polynomial \( Y = 0 \) which is irreducible over the integers. For example \( Y \) would be \( y^3 - x \) for a cube root of \( x \).

As discussed by Caviness and Fateman[1], multiple field extensions involving the same variable can be combined into a single field extension. For the purposes of integration, combine the field extensions involving the variable of integration \( x \) into a single variable \( y \) with its defining equation \( Y \).

In order to normalize polynomials with regard to \( Y \), each polynomial \( P \) containing \( y \) is replaced by \( \text{prem}(P,Y) \), the remainder of pseudo-division of \( P \) by \( Y \), as described by Knuth Volume 2[2].

While that process normalizes polynomials, it doesn’t normalize ratios of polynomials, for instance:

\[
1/y^2 = 1/(\sqrt[3]{x})^2 = \sqrt[3]{x}/x = y/x
\]

After the polynomials are normalized, if the denominator still contains the field extension \( y \), it is possible to move \( y \) to the numerator by multiplying both numerator and denominator by the \( y \)-conjugate of the denominator, then normalizing both numerator and denominator by \( Y \). The conjugate of a polynomial \( P \) with respect to \( Y \) can be computed by the following procedure where \( \text{deg}(q) \) is the degree of \( y \) in polynomial \( q \) and \( \text{pquo}(Y,P) \) and \( \text{prem}(Y,P) \) are the quotient and remainder of pseudo-division of \( Y \) by \( P \):

```python
def conj(P):
    if \text{deg}(P) < \text{deg}(Y):
        Q = \text{pquo}(Y,P)
        R = \text{prem}(Y,P)
    else:
        Q = 1
        R = 0
    if \text{deg}(R) == 0:
        return Q
    else:
        return Q \times \text{conj}(R)
```

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4. **Rational function integration with algebraic field extension**

With a single algebraic field extension $y$ which is a function of $x$, and the denominator free of $y$, and all the numerator factors in $p_1(x, y)$, the previous development can be reformulated:

$$f(x, y) = \prod_{j \leq 1} p_j(x, y)^j$$

(6)

The derivative of $f(x, y)$ with respect to $x$ is

$$\frac{df}{dx}(x, y) = \sum_{j \leq 1} j p_j(x, y)^{j-1} p_j'(x, y) \prod_{k \neq j} p_k(x, y)^k$$

(7)

Separating into numerator and denominator:

$$\frac{df}{dx}(x, y) = \sum_{j \leq 0} j p_j'(x, y) \prod_{k \neq j} p_k(x, y) = \frac{\sum_{j \leq 0} j p_j'(x, y) \prod_{k \neq j} p_k(x, y) p_1(x, y) + p_1'(x, y) \prod_{k \leq 0} p_k(x, y)}{\prod_{k \neq j} p_k(x, y)}$$

(8)

This time, $L$ is the whole numerator of equation (8). Note that the denominator includes $p_0(x, y)$; $p_0(x, y)$ does not contribute a term to $M$ because its coefficient $j$ is 0. Separating $p_1(x, y)$ from the denominator factors:

$$\sum_{j} j p_j'(x, y) \prod_{k \neq j} p_k(x, y) = \sum_{j \leq 0} j p_j'(x, y) \prod_{k \neq j} p_k(x, y) p_1(x, y) + p_1'(x, y) \prod_{k \leq 0} p_k(x, y)$$

(9)

Because they don’t involve $p_1(x, y)$, polynomials $M$ and $N$ can be computed from the square-free factorization of the denominator. The trick at each step is to construct a polynomial $t$ such that $M t + t' N$ cancels the highest term of dividend $R$ (initial $R = L$).

For polynomial $q$, $\deg(q'(x)) = \deg(q(x)) - 1$ in the previous section. The derivatives of polynomials involving $x$ and its algebraic field extension $y$ are more complicated. The derivative of $y$ is found by differentiating the $y$ defining equation $Y = 0$, then eliminating $y'$ from the chain rule for each term of the polynomial it occurs in. An example using the square of $y = (x^4 + a)^{1/3}$ demonstrates the reduction:

$$\frac{dy^2}{dx} = \frac{8 x^3}{3 (x^4 + a)^{1/3}} = \frac{8 x^3 (x^4 + a)^{2/3}}{3 (x^4 + a)} = \frac{8 x^3 y^2}{3 (x^4 + a)}$$

The degree of $y$ does not change between $y^2$ and $dy^2/dx$. However the degree of $x$ decreases; the degree of the denominator is one more than the degree of the numerator of $dy^2/dx$. This holds for any algebraic extension defined by a primitive polynomial.

Let $A$, $C$, and $R$ be rational expressions. Let $Q$ and $T$ be polynomials of $x$ containing no algebraic extensions of $x$. Let $g = \deg_x R - \deg_x N + 1$. The addition of 1 is to compensate for the reduction in the degree of $x$ in $p_1(x, y)$.

When there is no algebraic extension, $t = x^g$. If there is an algebraic extension $y$, let $q$ be the denominator of normalized $dy/dx$, $i$ be the integer quotient $g/\deg_x q$, and set $g$ to the remainder of $g/\deg_x q$. Then:

$$t = q^i x^g y^h$$

The polynomial $p_1(x, y)$ can be constructed by the following procedure given polynomials $L$, $M$, and $N$:

```python
A = 0
R = L
Q = denom(normalize(diff(y,x)))
Nyd = deg(N,y)
NyC = coeff(N,y,Nyd)
Nxd = deg(NyC,x)
```
while 0 < 1:
    Ryd = deg(num(R), y)
    RyC = coeff(num(R), y, Ryd)
    Rxd = deg(RyC, x)
    h = Ryd - Nyd
    g = (Rxd - Nxd + 1)
    if 0 == deg(Q, x):
        T = x^g
    else:
        i = quotient(g, deg(Q, x))
        g = remainder(g, deg(Q, x))
        T = Q^-i * x^g * y^h
    B = normalize( N*diff(T, x) + M*T )
    C = coeff(RyC, x, Rxd) * denom(B) / denom(R) / 
    coeff(coeff(num(B), y, Ryd), x, Rxd)
    A = A + C * T
    R = R - C * B
    if 0 == R:
        return A
    if deg(num(R), y) > Ryd:
        fail
    if deg(num(R), y) == Ryd and 
    deg(coeff(num(R), y, deg(num(R), y)), x) >= Rxd: 
        fail

The looping continues only as long as the degree of $R$ decreases. If this process succeeds, then the numerator of $A$ is $p_1(x, y)$; and the anti-derivative is $f(x, y) = A \prod_{j<1} p_j(x, y)^j$.

5. **Transcendental field extension**

Some transcendental field extensions involving a variable can be described by separable first-order differential equations:

\[
\frac{d}{dx} \log(x) = \frac{1}{x} \tag{10}
\]

Where $a$ is independent of $x$, applying the chain-rule:

\[
\frac{d}{dx} \log(x^a) = \frac{a}{x} \tag{11}
\]

Eliminating $x$ from equations (10) and (11) results in:

\[
\frac{d}{dx} \log(x^a) = a \frac{d}{dx} \log(x)
\]

Taking the anti-derivative of both sides:

\[
\log(x^a) = a \log(x) + C \tag{12}
\]

Thus $\log(x)$ is sufficient to be the canonical extension for both $\log(x)$ and $\log(x^a)$. Logarithm is multi-valued in the complex plane. Like radical functions, $\log(x)$ assumes one branch. The canonical form described here produces equations which are true for any branch of log assuming all references to $\log(x)$ are the same branch.

To describe the inverse function of $\log(x)$, substitute $y$ for $\log(x)$ and $\exp(y)$ for $x$:

\[
\frac{dy}{d\exp(y)} = \frac{1}{\exp(y)} \quad \frac{d}{dy} \exp(y) = \exp(y) \tag{13}
\]
Part of normalization for an expression (or equation) reduces the expression modulo the defining rules of the extensions appearing in that expression. For an algebraic extension this is its implicit defining equation (for example \((\sqrt{\pi})^2 = x\)). For a transcendental extension this is its defining differential equation.

These reductions serve to normalize expressions (involving variables) with unnested extensions. While (irreducible) algebraic expressions involving unnested transcendental extensions are canonical, when a transcendental function is composed with its inverse, an algebraic expression (without transcendental extensions) can result. The rest of this section addresses those nested transcendental expressions which reduce to algebraic forms.

For \(\log(\exp(x))\), the reduction results from integrating the combination of defining differential equations (14).

\[
\frac{\log(y)'}{y'} = \frac{1}{y} \frac{\exp(x)'}{x'} = \exp(x) \tag{14}
\]

\[
y = \exp(x) \quad \log(\exp(x))' = \frac{\exp(x)'}{\exp(x)} = x' \quad \log(\exp(x)) = x + C
\]

For \(\exp(\log(y))\), the combined equation is separated, integrated \((\ln = \log)\) is the result of the integration), and exponentiated:

\[
x = \log(y) \quad \frac{\exp(\log(y))'}{\exp(\log(y))} = \frac{y'}{y} \ln(\exp(\log(y))) = \ln(y) \quad \exp(\log(y)) = y
\]

The defining equations for arc-tangent and tangent are:

\[
\begin{align*}
\text{arctan}(x) &= \frac{x'}{x^2 + 1} \quad \frac{\tan(\theta)'}{\theta'} = \tan(\theta)^2 + 1 \\
x &= \tan(\theta) \quad \text{arctan}(\tan(\theta))' = \frac{\tan(\theta)'}{\tan(\theta)^2 + 1} = \theta' \quad \text{arctan}(\tan(\theta)) = \theta + C \\
\theta &= \text{arctan}(x) \quad \frac{\tan(\text{arctan}(x))'}{\text{arctan}(x)'} = \tan(\text{arctan}(x))^2 + 1 \quad \frac{\tan(\text{arctan}(x))'}{\tan(\text{arctan}(x))^2 + 1} = \frac{x'}{x^2 + 1}
\end{align*}
\]

For both \(\tan(\text{arctan}(x))\) and \(\exp(\log(x))\), composition through the defining equations results in a form:

\[
\frac{y'}{\Phi(y)} = \frac{x'}{\Phi(x)} \tag{15}
\]

Equation (16) is a stronger constraint than \(y' = x'\) in that it implies \(y = x\) without a constant of integration. Equation (16) is not the only form reducing to an algebraic relationship between \(y\) and \(x\). Equation (17) where \(b\) is a nonzero integer may also reduce to an algebraic relation:

\[
\frac{y'}{\Phi(y)} = \frac{bx'}{\Phi(x)} \tag{17}
\]

Separating \(y(x)\) into a ratio of relatively prime polynomials \(y(x) = f(x)/g(x)\):

\[
\frac{(f/g)'}{\Phi(f/g)} = \frac{f' g - f g'}{g^2 \Phi(f/g)} = \frac{bx'}{\Phi(x)} \quad (f' g - f g') \Phi(x) = b g^2 \Phi(f/g) x' \tag{18}
\]

If \(f' g - f g'\) is a polynomial, then \(g^2 \Phi(f/g) x'\) must also be a polynomial, which will only be the case when the denominator of \(\Phi(f/g)\) has degree 1 or 2 in \(x\). When \(g^2 \Phi(f/g) x'\) does not equal \((f' g - f g') \Phi(x)\), the scaled composition cannot be reduced to an algebraic relation.

For \(\Phi(y) = y\) and \(\Phi(y) = 1 \pm y\), it is the case that \(g^2 \Phi(f_b/g_b) = \Phi(x)^b\), and \(\Phi\) is related to \(y_b = f_b/g_b\) with \(b \geq 2\):

\[
(f_b/g_b - f_b g_b) \Phi(x) = b \Phi(x)^b x' \quad \rightarrow \quad f_b g_b - f_b g_b = b \Phi(x)^{b-1} x'
\]
In the case of $\Phi(y) = y$,

$$y_1 = x \quad y_b = y_{b-1} x \quad \frac{y'}{y} = \frac{b\ x'}{x} \rightarrow y = x^b$$

In the case $\Phi(y) = 1 + y^2$, using the sum-of-angles formula (a la Chebyshev):

$$\tan(b \theta) = \frac{\tan((b - 1) \theta) + \tan(\theta)}{1 - \tan((b - 1) \theta) \tan(\theta)}$$

Let $\theta = \arctan x$, $y_1 = x$, $y_b = f_b/g_b$:

$$y_b = \frac{y_{b-1} + x}{1 - y_{b-1} x} = \frac{f_{b-1} + g_{b-1} x}{g_{b-1} - f_{b-1} x} \quad y_{b-1} = \frac{y_{b} - 1 - x}{1 + y_{b} - 1 x} = \frac{g_{b} - 1 + f_{b} x}{g_{b} + f_{b} x}$$

Because $f_1 = x$ and $g_1 = 1$, the difference of the degrees of $f_b$ and $g_b$ alternate between 1 and $-1$ with the parity of $b$.

Field extensions for irreducible $\Phi$ polynomials will always reduce to identity when directly composed with their inverse function; those which can reduce scaled composition to more complicated algebraic expressions are limited to degrees of 1 or 2 by equation (18). Any $y_b$ recurrence must be symmetrical in $x$ and $y_{b-1}$. There are few symmetrical candidates for $y_b$ which might have transcendental-to-algebraic reductions. They were checked by formula (18) for $y_2(x) = f_2/g_2$.

For hyperbolic tangent $y_b = \tan(b \ atanh x)$, $\Phi(y) = 1 - y^2$, $y_1 = x$, $y = f/g$ and:

$$y_b = \frac{y_{b-1} + x}{1 + y_{b-1} x} = \frac{f_{b-1} + g_{b-1} x}{g_{b-1} + f_{b-1} x} \quad y_{b-1} = \frac{y_{b} - 1 - x}{1 - y_{b} - 1 x} = \frac{g_{b} - 1 - f_{b} x}{g_{b} - f_{b} x}$$

But $\Phi(x) = 1 - x^2$ is reducible, having factors $1 - x$ and $1 + x$; so it is not associated with a lone canonical field extension. Instead the integral uses field extensions $\log(x+1)$ and $\log(x-1)$:

$$\int \frac{dx}{1 - x^2} = \frac{\log(x+1) - \log(x-1)}{2}$$

Equation (16) is symmetrical. Both sides can be scaled by nonzero integers:

$$\frac{a\ y'}{\Phi(y)} = \frac{b\ x'}{\Phi(x)} \quad (19)$$

A composition of defining differential equations yielding a form (19) results in a relation $y_b = x_a$. For $\Phi(t) = t$, $y = (\sqrt{t})^b$. For other $\Phi$, the result is a polynomial relation $y_b = x_a$ which may or may not be solvable by radicals.

Because $x'/\Phi(x) + x'/\Phi(x) = 2 x'/\Phi(x)$, using $y_b$ without recurrence directs how to compose with a sum of transcendental functions.

$$\exp(\log x_1 + \log z_2) \rightarrow y = x_1 z_2$$

$$\tan(\arctan x_1 + \arctan x_2) \rightarrow y = (x_1 + x_2)/(1 - x_1 x_2)$$

A related problem is to normalize $\log(x z) \rightarrow \log x + \log z$. Taking the total differential of $\log(x z)$ yields:

$$\log'(x z) = \frac{x' z + x z'}{x z} = \frac{x'}{x} + \frac{z'}{z}$$

which is separable and integrable to $\log x + \log z$. This procedure works for $\arctan()$ as well:

$$\int \arctan \left( \frac{x + z}{1 - x z} \right)' = \int \frac{x'}{1 + x^2} + \frac{z'}{1 + z^2} = \arctan x + \arctan z$$
All trigonometric functions could have been canonically encoded with log() and exp() using imaginary transcendental constants ($i \pi$). But displaying formulas using the customary trigonometric functions would have been difficult.

These example nested transcendental functions simplify to rational functions (polynomial ratios):

\[
\begin{align*}
\exp(3 \log x) & \rightarrow \frac{y'}{y} = \frac{3x'}{x} \rightarrow y_3 = x^3 \\
\tan(4 \arctan x) & \rightarrow \frac{y'}{1+y^2} = \frac{4x'}{1+x^2} \rightarrow y_4 = \frac{4x - 4x^3}{1 - 6x^2 + x^4} \\
\tan(7 \arctan x) & \rightarrow \frac{y'}{1+y^2} = \frac{7x'}{1+x^2} \rightarrow y_7 = \frac{-7x + 35x^3 - 21x^5 + x^7}{-1 + 21x^2 - 35x^4 + 7x^6}
\end{align*}
\]

6. **Rational function integration with transcendental field extension**

For an integer power $w$ of an algebraic field extension $y(x)$:

\[
\deg_y \frac{dy^w}{dx} = w \quad \deg_x \frac{dy^w}{dx} = -1
\]

Transcendental field extensions behave differently:

\[
\begin{align*}
y = \exp(x^w) & \quad \deg_y \frac{dy}{dx} = 1 \quad \deg_x \frac{dy}{dx} = w - 1 \\
y = \log^w(x) & \quad \deg_y \frac{dy}{dx} = w - 1 \quad \deg_x \frac{dy}{dx} = -1
\end{align*}
\]

**Table 1** transcendental field extensions

References
