1   Motivating Question

Is $2 \in \pi$?

We want to say: "Invalid question!"

Set theory says: It depends on how you define 2 and π.

Type theory lets us say "invalid question." Types are kind-of like sets, except equality is a little strange. In particular, if you squint hard enough, proofs of equality look like paths in topological spaces. The goal of today's class is to convince you that proofs of equality look like paths.

2   Proving Equality

How can we prove equality? What are properties of equality?

- $x = x$ for all x (reflexivity)

- $x = y \rightarrow y = x$ for all x and y (symmetry)

- $x = y \rightarrow (y = z \rightarrow x = z)$ for all x, y, and z (transitivity)

- $x = y \rightarrow f\ x = f\ y$ for all x, y, and functions f (Leibniz property / substitution)

  N.B. We write $f\ x$ to denote the function f applied to x (traditionally written $f(x)$), because that is the notation that we will use in our proof assistant Coq.

3   Paths

What paths exist? What paths can we construct?

N.B. A path in a space X from A to B is a *directed* piece of (ideal, mathematical) string living in X.

- there is a path from A to A, for all points A (the constant path)

- given a path p from A to B, there is a path p^{-1} from B to A (inverse paths)

- given a path p from A to B and a path q from B to C, there is a path $p \cdot q$ from A to C (concatenation)

- given a path p from A to B in a space X, and a (continuous) function f from X to Y, there is a path from f A to f B, which we can denote f p.

- given a path p from A to B, the concatenation of p with its inverse, $p \cdot p^{-1}$, is in some sense "the same as" the constant path at A.

4　Equality of proofs of equality?

Can we find an analogue of this last property on the side of proofs of equality?

Consider the following two proofs:

Theorem. $1 = 1$

Proof 1. By reflexivity. □

Proof 2. We have a proof that $1 = 0 + 1$. (Exercise for the students.) Call this proof H. By symmetry on H, we have that $0 + 1 = 1$. By transitivity, we thus have $1 = 0 + 1 = 1$. □

This is a rather stupid proof. But so is the first one! In fact, they are stupidly the same proof. Let's prove this. To do so, we have to define equality.

5　Defining Equality

N.B. Our equality is homogeneous, which means that in order to ask the question "is x equal to y?" we must already have that x and y have the same type. If x and y have different types, the assertion $x = y$ is invalid (so the assertion $1 = $ apple is invalid).

Definition (Equality, version 1)**.**

　　There is a proof "By reflexivity" of the fact that $x = x$.[1]

($*$) Given x, y, and a proof that $x = y$, to prove any property of y, it suffices to prove that property of x.

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[1]More precisely, for any type A and any x of type A, there is a proof "By reflexivity at x" which proves that $x = x$.

With this definition, we can *prove* symmetry, transitivity, and substitution.

We use the notation $H \; : \; x = y$ to mean "H is a proof that $x = y$" or "H is of type $(x = y)$" or "H is an inhabitant of the type of proofs that $x = y$."

Theorem (symmetry). *Given x, y, and a proof $H : x = y$, we can prove $y = x$*

Proof.

- By ($*$), using the property $\square = x$, to prove $y = x$, it suffices to prove $x = x$

- have: $x = x$

    - by reflexivity

$\square$

Theorem (transitivity). *Given x, y, z, and a proof $H : x = y$, we can prove $y = z \rightarrow x = z$*

Proof.

- By ($*$), using the property $\square = z \rightarrow x = z$, to prove $y = z \rightarrow x = z$, it suffices to prove $x = z \rightarrow x = z$

- have $x = z \rightarrow x = z$

    - trivially

$\square$

Theorem (substitution). *Given x, y, a function f, and a proof $H : x = y$, we can prove $f \; x = f \; y$*

Proof.

- By ($*$), using the property $f \; x = f \; \square$, to prove $f \; x = f \; y$, it suffices to prove $f \; x = f \; x$

- have $f \; x = f \; x$

    - by reflexivity

$\square$

Let's rewrite a slight generalization of our two proofs that $1 = 1$, and prove them equal.
Suppose we are given x, y, and a proof $H : x = y$.

Theorem. $x = x$

Proof 1. By reflexivity.   □

Proof 2.

- have a proof $H_0 : x = y$

    - by H

- have a proof $H_1 : y = x$

    - by symmetry applied to H

- have $x = x$

    - by transitivity applied to H_0 and H_1

<div align="right">□</div>

Now let's prove that proof 1 and proof 2 are equal. To do this, we need a version of equality with a few more bells and whistles. Text that has been modified is underlined.

Definition (Equality, version 2)**.**

There is a proof "By reflexivity" of the fact that $x = x$.

($*'$) Given x, y, and a proof $\underline{H}$ that $x = y$, to prove any property of $\underline{y}$ and H, it suffices to prove that property of $\underline{x}$ and "by reflexivity".

Theorem. *Given x, y, and a proof $H : x = y$,*
(Proof 1 of x, y, and H) = (Proof 2 of x, y, and H)

Proof. By ($*'$), using the property "(Proof 1 of x, $\square_y$, and $\square_H$) = (Proof 2 of x, $\square_y$, and $\square_H$)", to prove this theorem, it suffices to prove that (Proof 1 of x, x, and "by reflexivity") = (Proof 2 of x, x, and "by reflexivity").

Let's plug this in to the body of the proof; we underline the values that we've plugged in.

Proof 2 on x, x, and "by reflexivity".

- have a proof $H_0 : x = \underline{x}$

    - <u>by reflexivity</u>

- have a proof $H_1 : \underline{x} = x$

    - by symmetry applied to reflexivity

- have $x = x$

    - by transitivity applied to H_0 and H_1

$\square$

Consider what this proof looks like as a path: We want to construct a path from x to x. First, we have a path, call it H_0, from x to x: let it be the constant path. Now, we have a path, call it H_1, from x to x: let it be the inverse of the constant path. Now concatenate H_0 and H_1 to get our path from x to x. This should really just be the constant path, trivially.

To prove this formally, we need one more property of equality.

Definition (Equality, version 3)**.**

There is a proof "By reflexivity" of the fact that $x = x$.

($*''$) Given x, y, and a proof $\underline{H}$ that $x = y$, to prove any property of $\underline{y}$ and $\underline{H}$, it suffices to prove that property of $\underline{x}$ and "by reflexivity".

(†) Given x, a property P, and a proof K that P is true of x and "by reflexivity", K is trivially equal to ($*''$) applied to x, x, "by reflexivity", the property P, and K.

By straightforward application of †, we can see the following:

Lemma. *Symmetry applied to "by reflexivity" is trivially "by reflexivity"*

If we further assume that the trivial proof that $x = x \to x = x$, when applied to "by reflexivity", gives "by reflexivity", then † also shows that

Lemma. *Transitivity applied to "by reflexivity" and "by reflexivity" is trivially "by reflexivity"*

With these two lemmas, it is straightforward to show that (Proof 1 of x, x, and "by reflexivity") is trivially equal to (Proof 2 of x, x, and "by reflexivity"), which is what we wanted to show. Thus, Proof 1 and Proof 2 are equal. $\square$

If these last few steps seem a bit hand-wavy, that's fine; we'll be doing them more formally in the proof assistant Coq tomorrow.