Computational Higher Inductive Types

Computing with Custom Equalities

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MIT CSAIL Student Workshop

April 10, 2014

Properties of Equality

- Reflexivity: $x = x$
- Symmetry: if $x = y$ then $y = x$
- Transitivity: if $x = y$ and $y = z$, then $x = z$
- Leibniz rule: if $x = y$, then $f(x) = f(y)$

Warm Up: Linked Lists

- Two constructors: `nil`, or `[]`, and `cons`
- Two accessors on non-nil lists: `head` and `tail`
- Equality is defined on an element-by-element basis
  - $[] = []$
  - $[] \neq [a, \ldots]$
  - $[a, \ldots] \neq []$
  - $[x_0, x_1, \ldots, x_n] = [y_0, y_1, \ldots, y_m]$ iff $[x_1, \ldots, x_n] = [y_1, \ldots, y_m]$ and $x_0 = y_0$
- Fairly easy to prove the properties of equality
  - In Coq, Agda, and Idris, you get all of these properties for free

Example: Unordered Sets

- `nil`, or $\emptyset$
- `add`
- `remove`
- `contains`
- Often implemented internally as a list or a tree
- Equality is then implemented as "is one a permutation of the other?"
- Fairly easy to prove that it's an equivalence relation
- Leibniz rule (if $x = y$, then $f(x) = f(y)$) is harder
- In Haskell, Agda, Coq, and Idris, the Leibniz rule is false! (or at least not internally provable)
  - The problem is that either you don't have private fields, or you can't make use of the fact that everything is defined in terms of your public methods.

Example: Unordered Sets

Solution 1: Canonical Inhabitants

- Give up private fields, but use element-wise equality
- Define a type of "sorted lists without duplication", and call them sets
- Now we can use element-wise equality, and get Leibniz (and other properties) for free
- What if we don't have an ordering on the elements, only equality?
- Is this really what we wanted? We asked for unordered sets, and instead made sorted lists.

Example: Unordered Sets

Solution 2: Higher Inductive Types

- Higher Inductive Types
- Keep the built-in equality (so we get the properties for free), but turn it into equality up to permutation
- How do we get that it's an equivalence relation for free?
  - Take the reflexive symmetric transitive closure of the given relation
- How do we get Leibniz for free?
  - Require proving it each time you define a particular function
  - To define a function that deals with unordered sets, you have to simultaneously prove that your function is invariant under permutations

Computing with Higher Inductive Types

- It seems simple enough, so what's the problem?
- Having higher inductive types gives you functional extensionality (if $f(x) = g(x)$ for all x, then $f = g$), which doesn't yet have a good computational interpretation in Coq nor Agda nor Idris
- Equality in Coq and Agda (`--without-K`) actually has a rich structure
- If you look at proofs of equality, and equality of these proofs, and you iterate this process, you get enough math to do topology!
- This is Homotopy Type Theory

Thank you

Thanks!

Questions?

Example: Unordered Sets

Solution 3: Parametricity

- Make use of the fact that private fields are private
- Very hard to do!
- Can probably be done by way of parametricity (aka "theorems for free"), or a generalization of it
- Parametricity can be given a computational interpretation, but it's very non-trivial to do so