Combinatorial Mathematics Notes

UIUC MATH 580, F'08

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Chapter 1

Combinatorial Arguments

1.1 Classical Models

Topic: Elementary Principles

Remark. ***Sum Principle***: If a finite set A is partitioned into sets $B_1, \ldots, B_k$, then $|A| = \sum_{i=1}^{k} |B_i|$. ***Product Principle***: If the elements of set A are built via successive choices, where the number of options for the ith choice is independent of the outcomes of the earlier choices, then $|A|$ is the product of the number of options for the successive choices.

Remark. ***Principle of Counting in Two Ways***: When two formulas count the same set, their values are equal.

Remark. ***Bijection Principle***: If there is a bijection from one set to another, then the two sets have the same size.

Remark. ***Pigeonhole Principle***: The maximum in a set of numbers is at least as large as the average (and the minimum is at least as small). In particular, placing more than kn objects into n boxes puts more than k objects into some box.

Remark. ***Polynomial Principle***: If two polynomials in x are equal for infinitely many values of x, then they are the same polynomial (and equal for all real x). The analogous statement holds with more variables by induction on the number of variables.

Topic: Words, Sets and Multisets

Definition 1.1.1 A ***k-word*** or ***word of length k*** is a list of k elements from a given set (the ***alphabet***); we may call the elements "letters". A ***simple word*** is a word whose letters are distinct. A ***k-set*** is a set with k elements; a k-set in a set S is a subset of S with k elements. We use $\binom{n}{k}$, read "n choose k", to denote the number of k-sets in an n-set. A ***multiset*** from a set S is a selection from S with repetition allowed.

Proposition 1.1.2 There are n^k words of length k from an alphabet S of size n. Equivalently, there are n^k functions from $[k]$ to S.

Definition 1.1.3 For $n \in \mathbb{N}_0$, we define the following products:

$$n \textbf{ \textit{factorial}}: n! = \prod_{i=0}^{n-1}(n-i).$$

$$\textbf{\textit{falling factorial}}: n_{(k)} = \prod_{i=0}^{k-1}(n-i).$$

$$\textbf{\textit{rising factorial}}: n^{(k)} = \prod_{i=0}^{k-1}(n+i).$$

Proposition 1.1.4 The number of simple k-words from an alphabet of size n is $n_{(k)}$.

Proposition 1.1.5 The number of k-element multisets from $[n]$, or equivalently the number of solutions to $\sum_{i=1}^{n} x_i = k$ in nonnegative integers is $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$.

Definition 1.1.6 A ***composition*** of the positive integer k is a list of positive integers summing to k. The entries in the list are the ***parts*** of the compositions.

Corollary 1.1.7 There are $\binom{k-1}{n-1}$ compositions of k with n parts.

Exercise 1.1.2. For each dice the probability of getting even/odd is 1/2. Then for first two dices, the probability of getting a total of even is again $1/4 + 1/4 = 1/2$. By induction, the probability is 1/2.

Exercise 1.1.3. For n, we need to count the possible distinct consecutive segments of length from 1 to n. There are $n + (n-1) + (n-2) + \ldots + 1$ of these. We do the same for m and multiply them to get the number.

Exercise 1.1.5. First problem is to pick four separators from 33 objects: $\binom{30+4-1}{4-1}$. For second one, we may assume that one candidate gets more than 15 votes. Then the rest splits 1 to 14 votes; there are $\sum_{i=0}^{14}\binom{i+3-1}{3-1}$ of outcomes. Subtracting these for each candidate from the total, we get the possible outcomes as $\binom{30+4-1}{4-1} - 4\sum_{i=0}^{14}\binom{i+3-1}{3-1}$.

Exercise 1.1.7. a) All the subsets, minus the ones with only even numbers... b) The k choices need at least $2k-1$ elements. When $n = 2k-1$, there is only a single choice. The choices are then $\binom{n-k+1}{k}$. c) The first one is just the product rule: one choice for $A_n = [n]$, then n choices for A_{n-1} etc. the answer is then $n!$. For the second question, we may first pick k elements from $[n]$ and assume A_n is these k elements. We then permute these elements and insert n bars to get A_{n-1} to A_0. Summing up these should give us the answer.

Exercise 1.1.10. We first pick 1 number from $1-9$ for the first digit. We then have two cases: 1. The chosen number is used again 2. The chosen number is not used again. We have

9 numbers left, and there are either 9^4 or 9^5 ways of constructing the rest. We exclude from these the cases where some number is used 3 or 4 times.

Exercise 1.1.14. This is simply counting all possible compositions for $k/2$.

Exercise 1.1.16.

Exercise 1.1.18. May treat the two disjointly; the runs separate m into $k-1$ pieces; it also puts k bars in n, with no two bars are consecutive.

Exercise 1.1.21. Every four points gives a pair of crossing chords, $\binom{n}{4}$.

1.2 Identities

Topic: Lattice Paths and Pascal's Triangle

Definition 1.2.1 A ***lattice path*** is a lattice walk in which each step increases one coordinate.

Remark. For lattice path of length n with height k, there are $\binom{n}{k}$ of these. We also obtain ***Pascal's Formula*** by noticing that there are two ways in the last step, one horizontal and one vertical. Therefore, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Theorem 1.2.2 (1.2.3 Elementary Identities) a

1) $\binom{n}{k} = \binom{n}{n-k}$

2) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

3) $k\binom{n}{k} = n\binom{n-1}{k-1}$ $\qquad \binom{k}{l}\binom{n}{k} = \binom{n}{l}\binom{n-l}{k-l}$

4) $\sum_k \binom{n}{k} = 2^n$ $\qquad \sum_k r^k\binom{n}{k} = (r+1)^n$

5) $\sum_{k=0}^{n} \binom{k}{r} = \binom{n+1}{r+1}$ $\qquad \sum_{k=-m}^{n} \binom{m+k}{r}\binom{n-k}{s} = \binom{m+n+1}{r+s+1}$

6) $\sum_k \binom{n}{k}^2 = \binom{2n}{n}$ $\qquad \sum_k \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$

Remark. ***Extended binomial coefficient*** $\binom{-u}{k} = (-1)^k\binom{u+k-1}{k}$.

Theorem 1.2.3 (1.2.6 Extended Binomial Theorem) For $u, x \in \mathbb{R}$ with $|x| < 1$, $(1+x)^u = \sum_{k\geqslant 0}\binom{u}{k}x^k$.

Topic: Delannoy Numbers

Definition 1.2.4 The Delannoy number $d_{m,n}$ is the number of paths from $(0,0)$ to (m,n) such that each move is by one of $\{(1,0),(0,1),(1,1)\}$. The number of the form $d_{n,n}$ are the ***central Delannoy numbers***.

Proposition 1.2.5 (1.2.10) The Delannoy number $d_{m,n} = \sum_k \binom{m}{k}\binom{n+k}{m}$.

Definition 1.2.6 The ***Hamming ball*** of radius m in n dimensions is the set consisting of the lattice points in $\mathbb{Z}^n$ that are within m unit coordinate steps from one point (such as the origin).

Proposition 1.2.7 The size of the Hamming ball of radius m in $\mathbb{Z}^n$ is $\sum_k \binom{n}{k}\binom{m}{k}2^k$.

PROOF. k denotes the number of non-zero coordinates $(\binom{n}{k})$; their absolute value sum to at most m, this is equivalent to $k+1$ positive integers summing to $m+1$ $(\binom{m}{k})$. The coordinates can be positive or negative (2^k). ■

Theorem 1.2.8 (1.2.13) For $m,n \in \mathbb{N}_0$, $\sum_k \binom{m}{k}\binom{n+k}{m} = \sum_k \binom{n}{k}\binom{m}{k}2^k$.

Exercise 1.2.2. Both are just choosing k from $m+n$.

Exercise 1.2.4. $\binom{m+k-1}{k} = \binom{m+k-1}{m-1}$; then applying Pascal's formula to get the sum. The other side is the same.

Exercise 1.2.6. Follow the formula...

Exercise 1.2.7. $(-1)^k$.

Exercise 1.2.8. Counting i^k by counting in two ways: one is the k-words from $[i]$, the other is number of repeated elements used. For example for i^2, if we use all different items, there are $\binom{i}{2}$ different ways of picking two elements; the ordering matters so $2\binom{i}{2}$ total. Then there are $\binom{i}{1}$ different ways of picking one element and use it twice (no order here). Therefore, $i^2 = 2\binom{i}{2} + \binom{i}{1}$.

Exercise 1.2.10. If two marbles are put in the same jar, it corresponds to a diagonal step; so it is easy to establish a bijection.

1.3 Applications

Topic: Graphs and Trees

Corollary 1.3.1 There are $2^{\binom{n}{2}}$ trees with vertex set $[n]$.

Definition 1.3.2 The ***functional digraph*** of a function $f : S \to S$ is the directed graph having vertex set S and an edge from x to $f(x)$ for each $x \in S$. There are $|S|$ edges, and each vertex is the tail exactly once.

Theorem 1.3.3 There are n^{n-2} trees with vertex set $[n]$.

Topic: Multinomial Coefficients

Proposition 1.3.4 (1.3.5) There are $m!/\prod_{i=1}^{n} k_i!$ words of length n having exactly k_i letters of type i, where $\sum_{i=1}^{n} k_i = m$.

Corollary 1.3.5 (1.3.6) The number of trees with vertex set $[n]$ in which vertices $1, \ldots, n$ have degrees $d_1, \ldots, d_n$, respectively, is $\frac{(n-2)!}{\prod (d_i-1)!}$.

Proposition 1.3.6 (1.3.8 Multinomial Theorem) For $n \in \mathbb{N}_0$, $\left(\sum_{i=1}^{n} x_i\right)^k = \sum \binom{k}{k_1, \ldots, k_n} \prod_{i=1}^{n} x_i^{k_i}$.

Topic: The Ballot Problem

Theorem 1.3.7 (1.3.12 Bertrand) Among the lists formed from a copies of A and b copies of B, there are $\binom{a+b}{a} - \binom{a+b}{a+1}$ such that every initial segment has at least as many As as Bs.

Lemma 1.3.8 (1.3.13) The central binomial coefficient $\binom{2n}{n}$ counts the following types of lattice paths of length $2n$ that start at $(0,0)$

A) Those ending at (n,n).

B) Those never rising above the line $y = x$.

C) Those never returning to the line $y = x$.

Theorem 1.3.9 (1.3.14) For $n \in \mathbb{N}_0, \sum_{k=0} n \binom{2k}{k}\binom{2n-2k}{n-k} = 4^n$.

Topic: Catalan Numbers

Definition 1.3.10 The ***Catalan sequence*** is defined by $C_n = \frac{1}{n+1}\binom{2n}{n}$ for $n \geqslant 0$. The number C_n is called the nth ***Catalan number***.

Theorem 1.3.11 (1.3.16) If p, q are relatively prime positive integers, then the number of lattice paths from $(0,0)$ to (p,q) that do not rise above the line $py = qx$ is $\frac{1}{p+q}\binom{p+q}{p}$.

Theorem 1.3.12 (1.3.21) The number of binary trees with $n+1$ leaves is the nth Catalan number, $\frac{1}{n+1}\binom{2n}{n}$.

Example 1.3.13 (1.3.22) There are $\frac{1}{n+1}\binom{2n}{n}$ triangulations of a convex $(n+2)$-gon.

Exercise 1.3.2. $n = \sum ik_i$. We may then first count the number of ways to partition n into m blocks of size ik_i (simply multinomial formula). Then for each ik_i, it is further partitioned into k_i blocks of size i (multinomial formula again). Multiplying gives us the result.

Exercise 1.3.3. The first row decides the second row; therefore we only need to work with the first row. We establish a bijection this way: suppose the numbers of the first row are $a_1, \ldots, a_n$. Let $k = a_{i+1} - a_i - 1$, we construct the lattice path by add one horizontal step followed by k vertical ones. This gives us a bijection (needs to add extra stuff).

Exercise 1.3.4. Catalan number over n factorial.

Exercise 1.3.5. Probability that the lattice path never goes above $y = x + 1$ after first step. Total paths this way is $\binom{a+b-1}{a-1} - \binom{a+b-1}{a}$. The denominator is again $\binom{a+b}{a}$.

Exercise 1.3.6. $4^n - 2\frac{1}{n}\binom{2n}{n}$: the total possible paths minus these that one never trails.

Chapter 2

Recurrence Relations

2.1 Obtaining Recurrences

Topic: Classical Examples

Example 2.1.1 Regions in the plane by n lines such that no three lines have a common point: $a_n = a_{n-1} + n$.

Example 2.1.2 Fibonacci numbers

Example 2.1.3 Derangements of n objects.

$D_n = n! - \sum_{k=1}^{n} \binom{n}{k} D_{n-k}$ for $n \geqslant 1$, with $D_0 = 1$.

$D_n = (n-1)(D_{n-1} + D_{n-2})$ for $n \geqslant 2$, with $D_0 = 1$ and $D_1 = 0$.

Example 2.1.4 Catalan number $C_n = \sum_{k=1}^{n} C_{k-1} c_{n-k}$, with $C_0 = 1$.

Topic: Variations

Example 2.1.5 Arrangements with n distinct available objects.

C(n, k) = C(n-1, k) + C(n-1, k-1) #k-subsets of $[n]$.

P(n, k) = P(n-1,k) + kP(n-1, k-1) # simple k-words from $[n]$.

S(n, k) = kS(n-1,k) + S(n-1, k-1) # partitions of $[n]$ with k blocks.

c(n, k) = (n-1)c(n-1, k) + c(n-1, k-1) # permutations of $[n]$ with k cycles.

Proposition 2.1.6 (2.1.10) The number of lattice points within m lattice steps of the origin in n-dimensional space satisfies the recurrence $a_{m,n} = a_{m,n-1} + a_{m-1,n-1} + a_{m-1,n}$ for $m, n > 0$.

Exercise 2.1.2. a_{n-1} gives us $n-1$ pairs. Adding p_{2n-1}, p_{2n} as a pair gives a pairing (a_{n-1} of these). Then we may pick a pair from the $n-1$ pairs and pair them with the last two persons ($2(n-1)a_{n-1}$ of these). Total is then $(2n-1)a_{n-1}$.

Exercise 2.1.5. $a_{n,k} = \binom{2n}{2k}/\binom{n}{k} \Rightarrow \binom{n}{k} = \binom{2n}{2k}/a_{n,k}$, so $\binom{n}{k-1} = \binom{2n}{2k-2}/a_{n,k-1}$, $\binom{n-1}{k-1} = \binom{2n-2}{2k-2}/a_{n-1,k-1}$. Therefore $\binom{2n}{2k}/a_{n,k} = \binom{2n}{2k-2}/a_{n,k-1} + \binom{2n-2}{2k-2}/a_{n-1,k-1}$.

Exercise 2.1.6. F_{n+1}.

Exercise 2.1.7. $a_n = 2a_{n-1} - a_{n-3}$. It would be $2a_{n-1}$ but we need to remove the of a_{n-1} that ends with 01. These are just a_{n-3} plus 01 at the end.

2.2 Elementary Solution Methods

Topic: The Characteristic Equation Method

Example 2.2.1 $F_n = F_{n-1} + F_{n-2} \Rightarrow x^2 - x - 1 = 0$. Solving this gives two roots a_1, a_2. Then $F_n = c_1 a_1^n + c_2 a_2^n$. Plugging in the initial condition gives us the constants.

Proposition 2.2.2 Consider a recurrence $a_n = (\sum_{i=1}^{k} c_i a_{n-i}) + f(n)$ such that $f(n) = F(n)c^n$, where F is a polynomial of degree d. If c has multiplicity r as a characteristic root of the homogeneous part (r may be 0), then the recurrence has a solution of the form $P(n)n^r c^n$, where P is a polynomial of degree at most d.

Topic: The Generating Function Method

Algorithm. The ***generating function method*** uses the following steps to solve a recurrence for $\langle a \rangle$.

1) Sum the recurrence over its "region of validity" (the values of the parameter where the recurrence holds) to introduce the generating function $A(x)$ and obtain an equation that $A(x)$ satisfies.

2) Solve this equation to express $A(x)$ in terms of x.

3) Find the formal power series expansion and set $a_n = [x^n]A(x)$.

Lemma 2.2.3 (2.2.17) For $k \in \mathbb{N}$, the power series expansion of $(1 - cx)^{-k}$ is

$$\frac{1}{(1-cx)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} c^n x^n.$$

Theorem 2.2.4 If $a_0 = 1$, and $a_n = \sum_{k=1}^{n} a_{k-1}a_{n-k}$ for $n \geqslant 1$, then $a_n = \frac{1}{n+1}\binom{2n}{n}$.

PROOF. Use the fact that the right side, when summed up, becomes $x[A(x)]^2$. The solve quadratic function of $A(x)$ to get the generating function. ■

Exercise 2.2.4. $1, 2$ are roots of characteristic equation. $f(n) = 1^n$, so $P(n)n^1 1^n$ is a special solution with $P(n)$ having degree 0. So special solution has form cn. Plugging in to recurrence to get $c = -1$.

Exercise 2.2.7. Cycle n has $2n - 2$ intersections with other cycles; each two consecutive points split the previous region into 2. So we have $a_n = a_{n-1} + 2n - 2 = \sum_{i=1}^{n-1} 2i = 2\binom{n}{2}$.

Exercise 2.2.8. $b_n = a_n + a_{n-1} = \lambda^n$. Then do $b_n - b_{n-1} + b_{n-2} \ldots$ to get a_n as a function of a_1 and λ.

Exercise 2.2.9. Proof via induction. Suppose $a_0, \ldots, a_{n-1}$ are integers, then $a_n - a_0 = \sum_{i=1}^{k} \lambda_i(\alpha^n - 1)$. It is easy to factorize the right and obtain that it is an integer.

Exercise 2.2.10. Straightforward verification.

2.3 Further Topics

Topic: The Substitution Method

Remark. ***Substitution Method*** basically substitute parts of recurrence to get a new recurrence that may be easier to handle. For example, $na_n = (n+1)a_{n-1}$ can be simplified by letting $a_n = (n+1)b_n$; we then have $b_n = b_{n-1}$.

Exercise 2.3.1. a) Let $b_n = (n+1)a_n$. b) Let $b_n = (3^n + 1)a_n$.

Exercise 2.3.2. a) Note that a_{2^k+1} to $a_{2^{k+1}}$ are all the same. b) Note that a_{2^k} to $a_{2^{k+1}-1}$ are all the same.

Chapter 3

Generating Functions

3.1 Ordinary Generating Functions

Topic: Modeling Counting Problems

Lemma 3.1.1 (3.1.7) $a_k = \sum_{j=0}^{k} b_j c_{k-j} \Rightarrow A(x) = B(x)C(x)$.

Topic: Permutation Statistics

Definition 3.1.2 For $\sigma \in S_n$, an ***inversion*** is a pair (σ_i, σ_j) such that $i < j$ and $\sigma_i > \sigma_j$.

Proposition 3.1.3 The enumerator of S_n by number of inversions is

$$(1+x)(1+x+x^2)\dots(1+x+x^2+\dots+x^{n-1}).$$

Proof. Adding 1 yields 0 inversion, so we have 1 for this. There are two ways to add 2 to 1, 12 does not create inversion, so it is 1; the other is 21 with one inversion, so we have x. 2 then contributes $(1+x)$ and is not affected by later additions; we then have the formula, with x^r indicates the number of inversions. ■

Definition 3.1.4 ***Canonical cycle representation*** lists cycles such that the first element of each cycle is the smallest of that cycle; then the cycles are listed in reverse order by the first elements. E.g. $(4731), (62), (89), (5)$ are listed as $(89)(5)(26)(1473)$. The representation can be written without the parentheses.

Lemma 3.1.5 (3.1.18) Canonical cycle representation is in bijection with permutations.

Theorem 3.1.6 (3.1.19) Let $c(n, k)$ be the number of elements of S_n with k cycles. The enumerator of S_n by number of cycles, $C_n(x)$, is given by

$$C_n(x) = \sum_{k=1}^{n} c(n,k)x^k = x^{(n)} = \prod_{i=1}^{n}(x+i-1).$$

Proof. Again do the insertion one by one. The x denotes the new cycle i will create, and the $i-1$ denotes the existing cycles it will keep. ■

Remark. Skipped Eulerian numbers.

Exercise 3.1.2. This is basically the same as 3.1.1, for each e_i, we have $s_i - r_i + 1$ of these, therefore the factor $\frac{1-x^{s_i-r_i+2}}{1-x}$. Hence $A(x) = \sum_{i=1}^{n} \frac{1-x^{s_i-r_i+2}}{1-x}$.

Exercise 3.1.4. $A(x) = (1 + x^2 + x^4 + \ldots)^2(1 + x + x^2 + \ldots)(1 + x^5 + x^10 + \ldots)$.

Exercise 3.1.5. $A(x) = (1 + x^2 + x^4 + \ldots)^2(1 + x + x^2 + \ldots)(1 + x^4 + x^8 + \ldots)$.

Exercise 3.1.6. $a_{n,k} = a_{n-1,k} + a_{n,k-1}$.

3.2 Coefficients and Applications

Remark. We may differentiate $A(x) = \ldots$ and the two sides are still equal. We can also do $A(B(x))' = A'(B(x))B'(x)$.

Topic: Operations and Summations

Remark. $\sum_{k\geqslant 0} \frac{1}{k+1}\binom{2k}{k}x^k = (1-\sqrt{1-4x})/(2x)$.

Example 3.2.1 $(1-x)^{-k} = \sum_n \binom{n+k-1}{k-1}x^n$ can be obtained by differentiate $\frac{1}{1-x} = \sum_n x^n$.

Proposition 3.2.2 (3.2.7) If A, B, C are the ordinary generating function for $\langle a\rangle, \langle b\rangle, \langle c\rangle$, respectively, then

$$\begin{array}{ll}
1) c_n = a_n + b_n \forall n & \Leftrightarrow C(x) = A(x) + B(x). \\
2) c_n = \sum_{i=0}^{n} a_i b_{n-i} \forall n & \Leftrightarrow C(x) = A(x)B(x). \\
3) b_n = \begin{cases} a_{n-k}, n \geqslant k \\ 0, n < k \end{cases} & \Leftrightarrow B(x) = x^k A(x) \\
4) b_n = n a_n & \Leftrightarrow B(x) = xA'(x) \\
5) c_n = \sum_{i=0}^{n} a_i & \Leftrightarrow C(x) = \frac{A(x)}{1-x} (\text{special case of 2)}) \\
6) b_n = \begin{cases} a_n, & n \text{ even} \\ 0, & \text{otherwise} \end{cases} & \Leftrightarrow B(x) = \frac{1}{2}[A(x) + A(-x)] \\
7) b_n = \begin{cases} a_n, & n \text{ odd} \\ 0, & \text{otherwise} \end{cases} & \Leftrightarrow B(x) = \frac{1}{2}[A(x) - A(-x)] \\
8) b_n = \begin{cases} a_k, n = mk \\ 0, k \nmid n \end{cases} & \Leftrightarrow B(x) = A(x^m)
\end{array}$$

Topic: Snake Oil

Remark. Snake Oil is used to evaluate sumes where the summand has several factors. Instead of evaluating the sum literally, we obtain a generating function of the sum and exchange the order of summation. Hopefully something nice will happen and we can get a generating function that is easy to handle.

Example 3.2.3 $\sum_{k\geqslant 0}\binom{k}{n-k}$. Let $a_n = \sum_{k\geqslant 0}\binom{k}{n-k}$, then $A(x) = \sum_n a_n = \sum_{k\geqslant 0} x^k \sum_n \binom{k}{n-k} x^{n-k} = \sum_{k\geqslant 0} x^k(1+x)^k = \frac{1}{1-x-x^2}$.

Exercise 3.2.2. For even k, $\frac{1}{2}[(1+x^2)^n + (1-x^2)^n]$ gives what we need. for odd k, $\frac{x}{2}[(1+x^2)^n + (1-x^2)^n]$ will do. So the answer is $\frac{1+x}{2}[(1+x^2)^n + (1-x^2)^n]$.

Exercise 3.2.3. $C(x) = \sum_{k\geqslant 0}\frac{1}{k+1}\binom{2k}{k}x^k = (1-\sqrt{1-4x})/(2x)$. So $A(x) = C(x)^2 = (1 - 2\sqrt{1-4x} + 1 - 4x)/(4x^2) = (1-\sqrt{1-4x}-2x)/(2x^2)$. Only $-\sqrt{1-4x}/(2x^2)$ actually counts; $a_n = \frac{1}{n+2}\binom{2n+2}{n+1}$.

3.3 Exponential Generating Functions

Definition 3.3.1 The ***exponential generating function (EFG)*** for a sequnce $\langle a\rangle$ is $\sum a_n x^n/n!$.

Topic: Modeling Labeled Products

Lemma 3.3.2 EFG $C(x) = A(x)B(x)$ if and only if $c_n = \sum_{j=0}^{n}\binom{n}{j}a_j b_{n-j}$.

Topic: The Exponential Formula

Remark. General and component structures. If the component structure can be expressed as an EFG $C(x)$ with $C(0) = 0$ then the general structure has $G(x) = e^{C(x)}$. For example, let a_n be the number of permutations of $[n]$ such that every cycle has odd length. This is the general structure. The component structure is then the case when $[n]$ itself is an odd cycle. This is only possible when n is odd and the number is $(n-1)!$. $C(x) = 1 + 2!\frac{x^3}{3!} + 4!\frac{x^5}{5!} + \ldots$. $G(x) = e^{C(x)}$.

Definition 3.3.3 The ***Sterling number*** $S(n,k)$ (of the "'second kind"') is the number of partitions of $[n]$ into k (nonempty) blocks. The Sterling number of first kind, $s(n,k)$, is $(-1)^{n-k}c(n,k)$, in which $c(n,k)$ is the number of permutations of $[n]$ with k cycles.

Exercise 3.3.2. Fix $n = 2p$ and we have $b_m = \sum_k \binom{m}{k} c_k d_{m-k}$, where $C(x)$ is the EFG of the sublists with odd numbers and $D(x)$ that of the even numbers. c_k counts the number of words of length k with alphabets from p, with each alphabet used and odd number of times. The EFG is $C(x) = (\frac{x}{2}(e^x - e^{-x}))^p$. Similarly we can get $D(x)$ then $B(x)$.

Exercise 3.3.5. For a single box, there is a single way of putting the objects in. The generating function is then $x^m e^x$. The general case is simply $(x^m e^x)^k$.

Exercise 3.3.6. Multiply both sides of a) with $S(m,n)$ and sum over n, we get $\sum_n S(m,n)a_n = \sum_n \sum_k S(m,n)s(n,k)b_k = \sum_k b_k \sum_n S(m,n)s(n,k) = \sum_k b_k \delta_{m,k}$ by 3.3.15. $\delta_{m,k} = 1$ only when $k = m$ so $\sum_k b_k \delta_{m,k} = b_m$; $b_m = \sum_n S(m,n)a_n$. Let $n = m, n = k$, we get 2).

Exercise 3.3.8. For a single team, we have $c_n = n$ since that is the number of ways to get the leaders for a single team of n members. Then the EFG of c_n, $C(x) = x(e^x)' = xe^x$ and $G(x) = e^{C(x)} = e^{xe^x}$.

3.4 Partition of Integers

Topic: Generating Function Methods

Theorem 3.4.1 The OGFs for partitions using parts in $\{1, \ldots, k\}$, paritions with largest part k, and all partitions are, respectively,

$$\prod_{i=1}^{k} \frac{1}{1-x^i} \qquad x^k \prod_{i=1}^{k} \frac{1}{1-x^i} \qquad \prod_{i=1}^{\infty} \frac{1}{1-x^i}$$

Topic: Ferrers Digrams

Proposition 3.4.2 The number of partitions of n with largest part k equals the number of partitions of n into k parts.

Exercise 3.4.1. The OFG is $A(x) = \frac{1}{1-x}\frac{x^3}{1-x^6} = \frac{x^3(1+x+x^2+x^3+x^4+x^5)}{(1-x^6)^2}$. Then $[x^3 0]A(x) = [x^2 4]\frac{1}{(1-x^6)^2}$. Since $a_n = \binom{n+1}{1}$, we need a_4, which is 5.

Chapter 4

Further Topics

4.1 Principle of Inclusion-Exclusion

Topic: PIE

Principle of Inclusion-Exclusion is the most common ***"Sieve Methods"***, which are counting methods that within a universe allow only a smaller desired set to survive a process of overcounting and undercounting.

Definition 4.1.1 :
U : The universe of all elements $\{x\}$.
$[n]$: $\{1, 2, \ldots, n\}$.
A_i : Subset of U with $1 \leqslant i \leqslant n$.
S, T : Subsets of $[n]$
$R(x)$: The **usage set** of x defined as: $\{i \in [n] : x \in A_i\}$. Note: this implies that x is not in the rest of A_i's. So $R(x) = \{1, 3\}$ with $[n] = \{1, 2, 3\}$ means $x \in (A_1 \cap A_3) \backslash A_2$
$f(S)$: $|\ \{x \in U : R(x) = S\}\ |$.
$g(S)$: $|\ \cap_{i \in S} A_i\ |$

For S as a subset of $[n]$, $f(S)$ counts the elements x whose usage set is fixed to S; $g(S)$ counts the elements whose usage set is a superset of S. We have $g(S) = \sum_{T \supseteq S} f(T)$; this can be visualized by looking at the ***Venn diagram***. $f(S) \leqslant g(S)$.

Theorem 4.1.2 Inclusion-Exclusion Principle, PIE. Let $A_1, A_2, \ldots, A_n$, be subsets of a universe U. With f and g as in above definition, the formula for f in terms of g is

$$f(T) = \sum_{S \supseteq T} (-1)^{|S|-|T|} g(S) = \sum_{S \supseteq T} (-1)^{|S|-|T|}\ |\ \cap_{i \in S} A_i\ |$$

To prove it, we note that the elements counted by $f(T)$ are counted only once in the formula by $g(S)$ when $S = T$ (can visualize through Venn diagram). For all the rest that are counted in the formula, they are counted the equal number of times in these $g(S)$ where $|S| - |T|$ is even and odd. Therefore, the terms all cancel out.

Application. Obtaining $f(T)$.

Topic: restricted permutations, rook polynomial $r_k(B)$

Application. Derangements. The formula for D_n, the number of derangements of $[n]$, is

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

We count this as permutations with no fixed points, which is

$$f(\varnothing) = \sum_{k=0}^{n} (-1)^n \binom{n}{k} (n-k)! = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

here $T = \varnothing$, $g(S) = \binom{n}{k}(n-k)!$.

Theorem 4.1.3 If f_p is the number of items having exactly p of n properties defined on a universe u, then

$$\sum_{p=0}^{n} f_p x^p = \sum_{k=0}^{n} (x-1)^k \sum_{|S|=k} g(S)$$

The left side counts each such element a with p properties exactly once in x^p. On the left side, $x^p = (x-1+1)^p = \sum_{k=0}^{n} (x-1)^k \binom{p}{k}$. a, with exactly p properties, satisfying $\binom{p}{k}$ k-sets of properties and therefore are counted that many times in $\sum_{|S|=k} g(S)$. For any fixed k, all these $\binom{p}{k}$ adds up to $\sum_{|S|=k} g(S)$.

Remark. For permutations with restricted positions, with $\sum_{|S|=k} g(S) = r_k(B)(n-k)!$, $\sum_{p=0}^{n} f_p x^p = \sum_{k=0}^{n} (x-1)^k r_k (n-k)!$. If we are to count the permutations with 0 properties (0 forbidden positions), then we let $x = 0$.

Application. The theorem can be used to calculate the number of elements satisfying exactly p properties. For example, if we have three subsets A_1, A_2, and A_3, and want to calculate the number of elements with two properties. **PIE** can only effectively gives us part of that sum. The textbook gives several examples (4.1.20, 4.1.21), one of which involves forbidden positions and rook polynomial.

Topic: singed involutions

Definition 4.1.4 An ***involution*** is a permutation whose square is the identity; the cycles have length 1 or 2. With respect of partition the universe X into a positive part X^+ and a negative part X^-, a ***signed involution*** on X ($n = |X|$) is an involution τ such that every 2-cycle pairs a positive element with a negative elements. Let F_τ and G_τ denote the sets of fixed points within X^+ and X^- under a signed involution τ.

A simple proposition says that if we let $w(x) = \pm 1$ for $x \in X^\pm$, then $|F_\tau| - |G_\tau| = \sum_{x \in X} w(x)$

Application. To use signed involution, usually we need to embed the desired set in a larger set X. For elements in X, we define a "switch" operation that keeps the desired set unchanged while switching the rest.

Topic: path systems

Application. To calculate a matrix determinant, we construct an acyclic digraph in which the sum of weights of all paths from x_i to y_j corresponds to the (i, j) entry in the matrix. The determinant is then equal to the sum of signed weights of disjoint paths systems.

Theorem 4.1.5 (4.1.31) Let X and Y be n-sets of vertices in a finite acyclic digraph G. If A is the X, Y-path matrix, and P is the set of disjoint X, Y-path systems, then

$$\det A = \sum_{p \in P} (\text{sign}(\sigma_p)) w(p)$$

Exercise 4.1.1. Third questions, let A_i be the set with no i in the n-tuple, then $10^n - \cup_{i=1,2,3} A_i$ is what we want. This is given by (using PIE) $10^n - 39^n + 38^n - 7^n$.

Exercise 4.1.2. The total number of outcome is 36^n. Let A_i be the set with no ii as a tossing outcome, then we can get the count of all "'ii"' happening by PIE.

Exercise 4.1.4. $252 = 2^2 3^3 7$. Numbers containing no such factors are coprime with 252. Let $f = 2, 3, 7$ and A_i be the set of positive integers less than 252 having f_i as a factor ($252/f_i - 1$ of these), applying PIE will give us the result.

Exercise 4.1.6. Total of 5 types of coin; let A_i be set that coin type i is used 5 or more times.

4.2 Polya-Redfield Counting

Topic: Burnsides' Lemma, the Pattern Inventory

Remark. The result from Burnside's Lemma is essentially that, for coloring of a set X with k colors, we look at the possible permutations of X based on X's structure. For example, for a 4-bead necklace, we may treat it as a square in the plane, with 8 symmetries: $\{(1)(2)(3)(4), (1234), (13)(24), (1432), (14)(23), (12)(34), (1)(3)(24), (2)(4)(13)\}$. For each i cycle we replace it with x_i, so we have $\frac{1}{8}(x_1^4 + x_4 + x_2^2 + x_4 + x_2^2 + x_2^2 + x_1^2 x_2 + x_1^2 x_2) = \frac{1}{8}(x_1^4 + 2x_4 + 3x_2^2 + 2x_1^2 x_2)$. If we want to get the possible colorings, we may simply replace all x_i with k. If we want to get more, for example, to color with two colors, r and b, we can replace each x_i with $r^i + b^i$ and gather desired terms. Say, if we want the number of colorings with two r and two b, then we colloect all coefficients of terms with r^2b^2.

Remark. For a rotating triangle, there are three possible outcome, $\{(1)(2)(3), (123), (132)\}$. So $f(x) = \frac{1}{3}(x_1^3 + 2x_3)$.

Exercise 4.2.1. $\frac{1}{3}(x_1^{15} + x_3^5)$? Since we still have the three rotations with one being identity, and each non indentity rotation has five 3-cycles.

Exercise 4.2.2. Similar as 4.2.1. The reason why the result is not a squre of putting flowers on the corner is that the rotations are related, not orthogonal.

Exercise 4.2.4. Same as 4.2.1.

Exercise 4.2.5. The key is that n is odd, so it has $n-1$ rotations, 1 identity, and n reflections. $f(x) = \frac{1}{2n}(x_1^n + (n-1)x_n + nx_1x_2^{(n-1)/2})$. Replace with k gives us the answer.

Exercise 4.2.6. Similar.

Chapter 5

5, 6, 7 Graph Intro

$|[S, \overline{S}]| < \delta(G) \Rightarrow |S| > \delta(G)$.
$\text{diag}(G) = 2 \Rightarrow \kappa'(G) = \delta(G)$.

Definition 5.0.1 A ***bond*** of a graph is a minimal nonempty edge cut.

Definition 5.0.2 A ***block*** of a graph G is a maximal connected subgraphp of G that has no cut-vertex.

Definition 5.0.3 $\kappa(x, y)$: Minimum size of an x, y-separating set.

$\kappa'(x, y)$: Minimum size of an x, y-edge separating set.

$\lambda(x, y)$: Maximum size of a set of independent x, y-paths.

$\lambda'(x, y)$: Maximum size of a set of edge disjoint x, y-paths.

Theorem 5.0.4 (7.2.5) Mininum X, Y-barrier has same size as maximum X, Y-link.

Theorem 5.0.5 (7.2.7) $\kappa(x, y) = \lambda(x, y)$ when $xy \notin G$, $\kappa'(x, y) = \lambda'(x, y)$ always.

Theorem 5.0.6 (7.2.9) G is k-connected if and only if $\lambda(x, y) \geqslant k$ for all $x, y \in V(G)$, and G is k-edge-connected if and only if $\lambda'(x, y) \geqslant k$ for all $x, y \in V(G)$.

Corollary 5.0.7 (7.2.10) $\kappa(G) = \kappa'(G)$ when $\Delta(G) = 3$.

Proof. $\kappa(G) \leqslant \kappa'(G)$ and there cannot be two paths that are not disjoint when $\Delta(G) = 3$. ■

Chapter 9

Planar Graphs

10/27/2008

9.1 Embeddings and Euler's Formula

Topic: Drawings and Duals

Proposition 9.1.1 $K_5, K_{3,3}$ are not planar.

Proposition 9.1.2 If $l(F_i)$ denotes the length of face F_i in a plane multigraph G with m edges, then $2m = \sum l(F_i)$.

Remark. For a planar graph G and its dual G^*, the dual of G^* is G if and only if G has a single component.

Theorem 9.1.3 Edges in a plane multigraph G form a cycle in G if and only if the corresponding dual edges form a bond in G^*.

Theorem 9.1.4 TFAE for a plane multigraph G:

- G is bipartite.
- Every face of G has event length.
- The dual G^* is Eulerian.

Definition 9.1.5 ***Outerplanar graph*** is a planar graph for which an embedding has all the vertices on the unbounded face.

Proposition 9.1.6 The boundary of the outer face of a 2-connected outerplanar multigraph is a spanning cycle.

Proposition 9.1.7 Every simple outerplanar graph has a pair of vertices of degree at most 2.

Topic: Euler's Formula

Theorem 9.1.8 (Euler's Formula) If a connected plane multi graph G has n vertices, m edges, and f faces, then $n - m + f = 2$.

Remark. Since every face in a plane graph must have at least 3 edges ($l(F) \geqslant 3$), $3f \leqslant 2m$. combining this with Euler's formula, we further have the following.

Theorem 9.1.9 Let G be a planar n-vertex graph with m edges. If $n \geqslant 3$, then $m \leqslant 3n - 6$. If G is triangle free, then $m \leqslant 2n - 4$.

Proposition 9.1.10 For an n-vertex plan graph G, the following are equivalent.

- G has $3n - 6$ edges.
- G is a triangulation.
- G is a maximal plane graph.

10/29/2008

9.2 Structure of planar Graphs

Topic: Kuratowski's Theorem

Definition 9.2.1 A ***Kuratowski subgraph*** is a subdivision of K_5 or $K_{3,3}$.

K_5 and $K_{3,3}$ are not planar. Any graph that contains a Kuratowski subgraph cannot be planar as well; since it is impossible to embed these subgraphs on a plane. On the other hand, Any graph without Kuratowski subgraph is a planar graph. Proving this involves the following lemmas and theorems.

Lemma 9.2.2 For a planar graph, any face can be made the outer face of some planar embedding.

The proof is straightforward using embedding on a ball.

Lemma 9.2.3 Every minimal nonplanar graph is 2-connected.

PROOF Suppose there is a 1-cut, then some lobes must be planar, therefore making the graph not minimal. ■

Lemma 9.2.4 Let $\{x, y\}$ be a separating 2-set of G, let $G_1, \ldots, G_k$ be the $\{x, y\}$-lobes of G, and let $H_i = G_i \cup xy$, if G is nonplanar, then some H_i is nonplanar.

PROOF If all H_i are planar, then $G + xy$ will have a planar embedding; G then also have a planar embedding. ■

Lemma 9.2.5 If G is a graph with fewest edges among all nonplanar graphs without Kuratowski subgraphs, then G is 3-connected.

PROOF Suppose not then G is 2-connected by the second lemma above. Then by previous lemma, some $H_i = G_i + xy$ is nonplanar. H_i has fewer edges than G, then H_i contains a Kuratowski subgraph. But then using some x, y-path in other x, y-lobes, G must also have a Kuratowski subgraph, contradiction. ■

Lemma 9.2.6 If $G \cdot xy$ has a Kuratowski subgraph, then G has a Kuratowski subgraph.

PROOF Show by analyzing cases that recovering x, y in G from $G \cdot xy$ cannot avoid a Kuratowski subgraph. ■

Theorem 9.2.7 If G is a 3-connected graph containing no subdivison of K_5 or $K_{3,3}$, then G has a convex embedding in the plane with no three vertices on a line.

PROOF Prove via induction (from K_4). For induction step, if G is 3-connected with more than 5 vertices, then there exists a 3-contractible edge xy in G by Lemma 7.2.19 in the text. By previous lemma, G has no Kuratowski subgraph, hence $G \cdot xy$ has no Kuratowski subgraph. Induction hypothesis then gives a convex embedding of $G \cdot xy$. From this we can try to reconstruct G. The only possible case allows a new convex embedding; other cases will create Kuratowski subgraphs. ■

9.3 Coloring of Planar Graphs

Topic: 5-colorable and 5-choosable

Theorem 9.3.1 (Five Color Theorem) Every planar graph is 5-colorable.

Proof It is equivalent to prove that a no 6-critical planar graph exists. 6-critical graph G must have $\delta(G) \geqslant 5$ and planar graph must have $\delta(G) \leqslant 5$. We can then assume G has a degree 5 vertex with neighbors using all 5 colors in consecutive order. Denoting these neighbors v_1, v_2, v_3, v_4, v_5. Then we may switch color 1 to 3 on v_1 and make any necessary corrections. If v_3 needs correction, then there must be a $1, 3$ path connecting them; but then we can change color 2 to 4 on v_2 and never gets to v_4. Therefore, G cannot be 6-critical. ■

Theorem 9.3.2 (Thomassen [1995])Every planar graph is 5-choosable.

Topic: Discharging

Proposition 9.3.3 (Wernicke [1904]) Ever planar triangulation with minimum degree 5 contains an edge with degree $5, 5$ vertices or degree $5, 6$ vertices.

Proposition 9.3.4 (Discharging rules) Let $V(G)$ and $F(G)$ be the sets of vertices and faces in a plane graph G, and let $l(\alpha)$ be the length of face α. The following equalities hold for G.

$$\sum_{v\in V(G)}(d(v)-6)+\sum_{\alpha\in F(G)}(2l(\alpha)-6)=-12 \quad \text{vertex charging}$$
$$\sum_{v\in V(G)}(2d(v)-6)+\sum_{\alpha\in F(G)}(l(\alpha)-6)=-12 \quad \text{face charging}$$
$$\sum_{v\in V(G)}(d(v)-4)+\sum_{\alpha\in F(G)}(l(\alpha)-4)=-12 \quad \text{balanced charging}$$

Remark. In the first formula, the second sum will be zero if we assume that we have a triangulation; even the unbounded face must be a triangle. For other face sizes, we need to change the formula accordingly. In the second formula, 3-regular planar graph is assumed. In the third, both vertices and faces are being considered.

Theorem 9.3.5 (Cranston [2008]) For $k \geqslant 7$, if G is a plane graph G with $\Delta(G) \leqslant k$ in which no two 3-faces share an edge, then G has an edge with weight at most $k+2$.

Remark. See book for proof, which uses balanced charging.

Chapter 10

Ramsey Theory

11/03/2008

10.1 The Pigeonhole Principle

Topic: Classical Applications

Example 10.1.1 (10.1.5) Covering a complete graph with bipartite graphs. The answer is $\lceil \log_2 n \rceil$. Suppose that K_n is covered by bipartite subgraphs $G_1, \ldots, G_k$. Let X_i, Y_i be the bipartition of G_i. We may assume that each G_i contains all the vertices, since adding isolated vertices doesn't introduce odd cycles.

For each vertex v, define a binary k-tuple α by setting $\alpha_i = 0$ if $v \in X_i$ and $\alpha_i = 1$ if $v \in Y_i$. There is an uv edge if and only if the two k-tuples of u, v differ. Since there are 2^k different two tuples, K_{2^k+1} cannot be covered by k bipartite graphs. On the other hand, we may always arrange the bipartite graphs such that the vertices have different k-tuples, as long as we have no more than 2^k vertices.

Example 10.1.2 (10.1.7) Forcing divisible pairs. If S is a set of $n+1$ numbers in $[2n]$, then S contains two numbers such that one divides the other.

To apply the Pigeonhole Principle, we partition $[2n]$ into n classes such that for every two numbers in the same class, one divides the other. The classes are $(1, 2, 4, 8, \ldots), (3, 6, 12, 24, \ldots), (5, 10, 20, \ldots), \ldots$.

Example 10.1.3 (10.1.8) A domino tiling problem. A 6-by-6 checkerboard can be partitioned into 18 dominoes consisting of two squares each; this is a ***tiling*** of dominoes. We prove that every such tiling can be cut between two adjacent rows or adjacent columns without cutting any dominoes. That is, for any tiling, the board can be cut into two rectangular parts without cutting any dominoes.

To use the Pigeonhole Principle, we observe that there are 18 dominoes and 10 lines with which the board can be cut into two rectangular pieces. We further observe that any domino crossing a line must have another domino cross the same line, otherwise neither piece can be finished.

Theorem 10.1.4 (10.1.9) If T is spanning tree of the k-dimensional hypercube Q_k, then there is an edge of Q_k outside T that forms a cycle of at least length $2k$.

Use antipodal vertices of vertices of T and mark the first edge from a vertex of T. There are 2^k vertices but only $2^k - 1$ edges, therefore, some edge is marked twice. ...

Topic: Monotone Sublists

Example 10.1.5 (10.1.14) Every list of more than n^2 real numbers has a monotone sublist with length more than n.

Construct a list of (x_i, y_i) pairs for a_i such that one count the number of increasing lists a_i is in and the other counts the decreasing lists a_i is in. Every (x_i, y_i) then must be different.

Theorem 10.1.6 (10.1.17) If the $\binom{n}{2}$ edges of a complete graph on n vertices have the distinct labels $1, \ldots, \binom{n}{2}$, then some trail of length at least $n-1$ has an increasing list of labels.

Let the weight of a vertex be the length of the longest increasing trail ending there; if the total of weights is at least $n(n-1)$, the Pigeonhole Principle guarantees a increasing trail of length at least $n-1$.

To count the weight, we add edges according to their labels in increasing order. For first edge added, both end vertices get weight 1. For any edge added at some point, its two end vertices have weights either the same or different. If the weights are the same i, since the edge is increasing order, both vertices now have weight $i+1$. If the weights are different, say $i < j$, then the i becomes $j+1$ and the j stays the same. In either case, the total weights go up by 2. Since we have $\binom{n}{2}$ edges, we have total weight at least $n(n-1)$.

Example 10.1.7 (10.1.25) Graphs with girth 6 and high chromatic number.

The construction is inductive. It takes a G with girth at least 6, chromatic number $k-1$, and $|V(G)| = r$. Let $N = (r-1)(k-1)+1$, take a set S of N isolated vertices and for each $\binom{N}{r}$ vertices in the set, connecting these r vertices to the vertices of a copy of G through r

edges (there are $\binom{N}{r}$ copies of G). The construction keeps the girth to be 6 at the least and by the Pigeonhole Principle, there are r vertices in S colored with same color. Then the matching G of these r vertices in S cannot be colored using $k-1$ colors.

11/05/2008

10.2 Ramsey's Theorem

Topic: The Main Theorem

Example 10.2.1 Among any 6 people, there are three mutual acquaintances or three mutual strangers. This problem can be modeled as coloring of edges with red and blue of a K_6 and show that there exists a red K_3 or a blue K_3.

Definition 10.2.2 (10.2.2) A k-***coloring*** is a function that labels each domain element with one of k colors (typically $[k]$). We use $\binom{S}{r}$ to denote the family of r-subsets of a set S. Under a coloring of $\binom{S}{r}$, a set $T \subseteq S$ is ***homogeneous*** if its r-subsets all have the same color.

Remark. In the previous example, $S = 6, k = 2, r = 2$ and we color all the edges ($\binom{S}{r} = \binom{6}{2}$ of these). We are trying to find a set T with $|T| = 3$ for either color. If we allow different T for different colors, then we may have a set S with $k, r, p_1, p_2, \ldots, p_k$ s.t. we want to guarantee that we have at least one set T in which every p_i subset is colored with color i.

Definition 10.2.3 (10.2.3) In a k-coloring of $\binom{S}{r}$, a homogeneous set in which all r-sets have color i is i-***homogeneous***. Given quotas $p_1, \ldots, p_k \in \mathbb{N}$, if there exists $N \in N$ such that in every k-coloring of $\binom{N}{r}$ there is an i-homogeneous set of size p_i for some i, then the smallest such integer is the ***Ramsey number*** $R(p_1, \ldots, p_k; r)$.

Example 10.2.4 Extending the previous example, let's say we want a red K_p and a blue K_q. We want to show that for any p, q, there exists a smallest number $R(p, q; 2)$. We claim that

$$R(p, q; 2) \leqslant R(p-1, q; 2)+ \leqslant R(p, q-1; 2).$$

This is true because by the Pigeonhole Principle, in a complete graph with $R(p-1, q; 2)+ \leqslant R(p, q-1; 2)$ vertices, any vertex u have either at least $R(p-1, q; 2)$ red edges or at least $R(p, q; 2)$ blue edges. Suppose that u has $R(p-1, q; 2)$ red edges, then we have in the set $R(p-1, q; 2)$ either a blue K_q (we are done) or a red K_{p-1}. With a red K_{p-1}, adding red edges from u gives the red K_p. The other case is the same.

Theorem 10.2.5 (10.2.5 Ramsey) Given $k, r, p_1, \ldots, p_k \in \mathbb{N}$, the Ramsey number $R(p_1, \ldots, p_k; r)$ exists.

Proof. The proof use same reasoning as in the previous example. For two colors $p_1 = p, p_2 = q$, we pick a vertex u, we have to consider all the r subsets it is in. We want to guarantee that there is a subset T of $S - u$ such that all $r-1$ subsets in T plus u are colored the same (say red). If T is large enough to guarantee either a $p-1$ set with $r-1$ subsets red or a q set with r subsets blue, then we are good. So the final number is

$$N = 1 + R(p', q'; r-1), p' = R(p-1, q; r), q' = R(p, q-1; r),$$

that is, we pick u and we are guaranteed to have either set of size p' with all $r-1$ subsets colored red, or a set q' with all $r-1$ subsets colored blue. Rest is clear. ■

Remark. Ramsey Theory is a generalized version of the Pigeonhole Principle; when $r = 1$, it becomes the Pigeonhole Principle.

Theorem 10.2.6 (10.2.6) For $m \in \mathbb{N}$, there is a (least) integer $N(m)$ such that every set of at least $N(m)$ points in the plane (no three collinear) contains an m-subset forming a convex m-gon.

Proof. There are two facts: 1. Any 5 points contains a convex 4-gon. 2. m points in the plane form a convex m-gon if any $\binom{m}{4}$ points form a convex 4-gon.

Now if we let $N = R(m, 5; 4)$ points in the plane be colored such the 4 subsets are red if they are convex and blue if not. Then there either exists a m-set in which all 4 subsets are red (convex), or there is a 5-set with all 4 subsets blue (non-convex). But since we know that the second case is not possible, N guarantees a convex m-gon. ■

Definition 10.2.7 A ***storage strategy*** T assigns each n-set A of keys a storage permutation: if $T(A) = \sigma$, then the jth smallest element of A goes in location $\sigma(j)$, for $1 \leqslant j \leqslant n$. A ***query*** asks whether a key x is present in the table. A ***search strategy*** S probes successive locations based on x, T, and the outcome of earlier probes. The answer to a probe is the key stored there.

Lemma 10.2.8 (*10.2.8 Yao) Let T be a storage strategy for n-sets from a universe M with $|M| \geqslant 2$. A set $P \subseteq M$ is ***stored consistently*** if each n-set in P is stored according to the same permutation. If some set of size $2n-1$ in M is stored consistently under T, then the cost of T is at least $\lceil \lg(n+1) \rceil$ from every search strategy.

Theorem 10.2.9 (*10.2.9 Yao) Let $f(m, n)$ be the complexity of membership testing when n-sets from a space of m keys are stored in a table of size n. If m is sufficiently large, then $f(m, n) = \lceil \lg(n+1) \rceil$.

11/07/2008

Topic: Ramsey Numbers

Theorem 10.2.10 (10.2.11) $R(p,q) \leqslant R(p-1,q) + R(p,q-1)$. If both summands on the right are even, then the inequality is strict.

PROOF. The inequality obviously holds since in any $R(p-1,q) + R(p,q-1) - 1$ vertices, if we don't have $K_p, \overline{K}_q$, then there are either a K_{p-1} or $\overline{K}_{q-1}$. Adding the last vertex then gives us the result. Equality requires any $R(p-1,q) + R(p,q-1) - 1$ vertices to have no K_p and no $\overline{K}_q$. This in turn requires that every set of $R(p-1,q) + R(p,q-1) - 1$ vertices to all have degree $R(p-1,q) - 1$. But the total degree of $R(p-1,q) + R(p,q-1) - 1$ vertices is then $D = (R(p-1,q) + R(p,q-1) - 1)(R(p-1,q) - 1)$. If both summands are even as in the assumption then D is odd, which is not possible as a total degree of any graph. ■

Corollary 10.2.11 (10.2.11) $R(p,q) \leqslant \binom{p+q-2}{p-1}$.

PROOF. Obviously, $R(p,2) = p$. Applying previous theorem we can inductively prove the claim. ■

Theorem 10.2.12 (10.2.14, Erdos) $R(p,p) > \frac{1}{e\sqrt{2}} p 2^{p/2}(1 + 0(1))$.

PROOF. Counting argument is used in the proof. For each set of p vertices, if they form a clique, there are at most $2^{\binom{n}{2}-\binom{p}{2}}$ subgraphs containing the clique. Similarly, if they form an independent set, there are at most $2^{\binom{n}{2}-\binom{p}{2}}$ subgraphs containing the independent set. Since there are $\binom{n}{p}$ choices of p vertices in n vertices, there can be at most $2\binom{n}{p}2^{\binom{n}{2}-\binom{p}{2}}$ subgraphs containing p-cliques or independent sets of size p. If $2\binom{n}{p}2^{\binom{n}{2}-\binom{p}{2}} < 2^{\binom{n}{2}}$ for some n, then $R(p,p) > n$ since there are some subgraphs left without either p-clique or independent set of size p. Using an inequality $\binom{n}{p} < (ne/p)^p$ then gives the result. ■

Definition 10.2.13 For graphs $G_1, \ldots, G_k$, the ***(graph) Ramsey number*** $R(G_1, \ldots, G_k)$ is the smallest integer n such that every k-coloring of $E(K_n)$ contains a copy of G_i in color i for some i. When $G_i = G$ for all i, we write $R(G_1, \ldots, G_k)$ as $R_k(G)$.

Remark. Exercise 5.4.15 says that every graph G with minimum degree at least $m-1$ contains every tree of m vertices.

Theorem 10.2.14 (10.2.17) If T is an m-vertex tree, then $R(T, K_n) = (m-1)(n-1) + 1$.

PROOF. For lower bound, color $n-1$ copies of K_{m-1} red, and the rest edges between the K_{m-1}'s blue. No K_{m-1} can have red m vertex tree and the blue edges form a $n-1$-partite graph and does not contain a blue K_n.
For the upper bound, the proof is inductive on n, invoking a property of trees proved by induction on m. Base case K_1 is obvious. For $n > 1$, if 2-coloring of $K_{(m-1)(n-1)+1}$ has any vertex x with at least $(m-1)(n-2)+1$ blue edges, then induction hypothesis gives us a red T or a blue K_{n-2}. Adding x and we are done.
Otherwise, every vertex has at most $(m-1)(n-2)$ blue edges. This leaves $m-1$ edges for every vertex. By the remark, we can find any tree T of m vertices. ■

10.3 Further Topics

Topic: Ven der Waerden's Theorem

Theorem 10.3.1 (10.3.10, Schur) Given $k > 0$, there exists an integer s_k such that every k-coloring of the integers $1, \ldots, s_k$ yields monochromatic (but not necessarily distinct) x, y, z solving $x + y = z$.

PROOF. From k-coloring f of the integers, define f', the coloring of edges with vertices the integers from the k-coloring, as $f'(e_{i,j}) = f(|i-j|)$. We have that $R_k(3;2)$ will guarantee a 3-set with every 2-set colored the same; that is, a triangle of same color. Suppose the vertices of the triangle are integers $i < j < k$, then $x = j - i, y = k - j, z = k - i$ satisfy $x + y = z$ and they are colored the same in the original k-coloring. ■

Remark. A constructive lower bound of the minimum Schur number is $(3^k + 1)/2$; this is obtained as a recurrence by leaving enough space so that $x + y = z$ cannot have monochromatic solution.

Theorem 10.3.2 (10.3.14, Van der Waerden, 1927) Given positive integers l, k, there exists an integer $w(l,k)$ such that every k-coloring of $[w(l,k)]$ contains a monochromatic l-term arithemetic progression.

Remark. This theorem is not proved in class. Only the following example is proved.

Example 10.3.3 $w(3,2) \leqslant 325$. Partition the coloring into 65 blocks of 5 consecutive numbers; no matter how the first 5-block is colored, we can find 5-block colored the same way within 33 blocks since there are 32 ways to 2-color a 5-block. Say that block is block $1 + k$, then block $1, 1 + k, 1 + 2k$ are all within the 65 blocks. If the first elements are the same in the blocks, then we are done. Otherwise, we may assume the first element of first block is R and there are two cases: if the first three elements of first/second block are RRB (it cannot be

RRR), that will force third block to be BBR, then we have $\bar{R}RB, \ldots, R\bar{R}B, \ldots, BB\bar{R}$ as what we want. The other case is RBX, suppose $X = B$, then we have $RB\bar{B}, \ldots, R\bar{B}B, \ldots, \bar{B}BR$; if $X = R$, then last element of the block cannot be R (otherwise we have $\bar{R}B\bar{R}X\bar{R}$), we then have $\bar{R}BRXB, \ldots, RB\bar{R}XB, \ldots, BRBX\bar{R}$.

Chapter 12

Partially Ordered Sets

11/10/2008

12.1 Structure of Posets

Definition 12.1.1 An ***partial order*** relation is an R that is

> ***reflexive***: $xRx, \forall x$,
>
> ***antisymmetric***: $xRy, yRx \Rightarrow x = y$, and
>
> ***transitive***: $xRy, yRz \Rightarrow xRz$.

Definition 12.1.2 The ***comparability digraph*** of a poset P is the digraph that has an edge $x \to y, \forall x \leqslant y$. ***comparability graph*** is the digraph with orientation removed. We say y ***covers*** x if $x \leqslant y$ and no z exists such that $x \leqslant z \leqslant y$. The ***cover digraph*** and ***cover graph*** are the digraph and graph of vertices of P under cover relation.

Definition 12.1.3 A ***chain*** in a poset is a subset such that every two elements are comparable; hence a total order. A ***antichain*** is a subset such that no two elements are comparable. An element is ***maximal*** if no elements is greater than it and ***minimal*** if no element is smaller than it.

Definition 12.1.4 Let P be a poset, its ***width*** $w(P)$ is the size of the largest antichain in P. Its ***height*** $h(P)$ is the size of the largest chain in P. Its ***length*** is one less than its height.

Definition 12.1.5 A ***family*** in a poset P is a subset of P. An ***(order) ideal*** or ***down-set*** is a family I of P such that $x \in I, y \leqslant x \Rightarrow y \in I$. The ***dual ideal*** or ***up-set*** is the an I such that $x \in I, x \leqslant y \Rightarrow y \in I$.

Theorem 12.1.6 (12.1.17, Dilworth) If P is a finite poset, then the maximum size of an antichain in P equals the minimum number of chains needed to cover the elements of P.

PROOF. First we see that $w(P)$, the maximum size of an antichain, is at most the minimum number of chains that is needed for a cover of P since if there are more, then some two elements must be in some chain and cannot be in an anitchain. Therefore, all we need to prove is to find one antichain from a set of minimum number of chains that cover P. See book for rest of proof. ■

11/12/2008

Remark. It is worth noting that chains may skip elements in between. Suppose that we have a total order $a < b < c < d < e < f$, then ace and bdf are both chains.

Theorem 12.1.7 (12.1.18, Fuklerson) Dilworth's Theorem is equivalent to the Konig-Egervary Theorem on matching in bipartite graphs: the maximum size of a matching equals the minimum size of a vertex cover.

PROOF. ($\Rightarrow$) View a bipartite graph as a poset with one partite set as maximal elements and the other minimal. The maximum chain has length 2 (a single edge). Every chain cover of the poset of size $n - k$ then uses k chains of length 2 and the rest are simply vertices ($n - 2k$ of these). These k chains is then a matching. Each antichain is an independent set in the graph since there cannot be edges between any two elements in an antichain. That is, an independent set of size $n - k$ then leaves k vertices in the bipartite graph that forms a vertex cover. Applying Dilworth's Theorem then gives us that maximum size of antichain (minimum vertex cover) equals the minimum number of chains (maximum matching).

($\Leftarrow$) Turn a poset into a bipartite graph: make two copies of each element in the poset; a $+$ copy and a $-$ copy and put them into a $+$ partite set and a $-$ partite set. There is an edge between x^-, y^+ if $x \leqslant y$ in the poset. The rest is then basically the reverse of the ($\Rightarrow$) part. ■

12.2 Symmetric Chains and LYM

Topic: Ranked and Graded posets

Definition 12.2.1 A function $r : P \to \mathbb{Z}$ is a ***rank function*** on P if $r(y) = r(x)+1$ whenever y covers x. A poset with a ranked function is a ***ranked poset***. A poset P is ***graded*** if all its maximal chains have the same length, and its ***rank*** $r(P)$ is that length. The ***height*** $h(x)$ of an element x is the maximum length of a chain in P having x as its top element.

Definition 12.2.2 If P is graded, then the elements with rank k are the kth ***rank*** or kth ***level*** P_k, and we write $N_k(P)$ for the ***rank size*** $|P_k|$. A graded poset is ***rank symmetric*** if $N_k = N_{r(P)-k}$ for all k. It is ***rank unimodal*** if there is a rank k such that $N_i \leqslant N_j$ whenever $i \leqslant j \leqslant k$ or $i \geqslant j \geqslant k$. The ***rank generating function*** is the formal power series $\sum_{k\geqslant 0} N_k x^k$.

Example 12.2.3 $\mathbf{2}^n$ is the poset formed with elements of subsets of $[n]$ and inclusion order relation. It is graded since every longest chain contains every size subset from $0 \to n$. For any element a, $r(a) = |a|$, the size of the set. $r(\mathbf{2}^n) = n$. The kth rank is $\binom{[n]}{k}$; and $N_k = \binom{n}{k}$. The rank generating function is $(1+x)^n$. The poset is rank-symmetric and rank-unimodal.

Example 12.2.4 Divisors of an integer, or multisets. The divisor of a positive integer N form a poset $D(N)$ under divisibility. It is graded and rank-symmetric.

Topic: Symmetric Chain Decomposition

Definition 12.2.5 A chain in a graded poset P is ***symmetric*** if it has an element of rank $r(P) - k$ whenever it has an element of rank k. A chain is ***consecutive*** or ***skipless*** if its elements lie in consecutive ranks. A ***symmetric chain decomposition*** of P is a partition of P into symmetric skipless chains. A poset with a symmetric chain decomposition is a ***symmetric chain order***.

Example 12.2.6 Since every chain in a symmetric chain decomposition intersects the antichain formed by the middle rank, every symmetric chain decomposition is a Dilworth decomposition (a decomposition into the minimum number of chains). Since every chain is symmetric and skipless, a symmetric chain order must be rank-symmetric and rank-unimodal.

Theorem 12.2.7 (12.2.9) $\mathbf{2}^n$ is a symmetric chain order.

PROOF. Via inductive construction over n. ∎

Theorem 12.2.8 (12.2.10) Products of symmetric chain orders are symmetric chain orders.

Example 12.2.9 Bracketing decomposition of $\mathbf{2}^n$. For a subset $A \subset [n]$, complete a n digit binary number by writing a 1 at ith position if $i \in A$ and 0 otherwise. Then turn each 0 into a left bracket and each 1 into a right bracket. After we get the brackets, we mark the "matched" brackets and may call these ***fixed*** brackets. Any two bracketing structures with the same fixed brackets belong to the same chain. See page 675 for an example.

Theorem 12.2.10 (12.2.12) The inductive and bracketing decomposition of $\mathbf{2}^n$ are the same.

Theorem 12.2.11 (12.2.15) The number of monotone Boolean functions is at most $3^{\binom{n}{\lceil n/2 \rceil}}$.

11/14/2008

Topic: LYM and Sperner Properties

Definition 12.2.12 A ***k-family*** in a poset P is a family with no chains of size $k+1$. A graded poset has the ***Sperner property*** if its largest-sized rank is a maximum antichain. It has the ***strong Sperner Property*** if for all k its k largest ranks form a maximum k-family.

Theorem 12.2.13 (12.2.16) In $\mathbf{2}^n$, the elements of rank $\lfloor n/2 \rfloor$ form a maximum antichain.

PROOF. The middle rank elements obviously form an antichain (since these sets are all of the same size, and are pairwise different); we only need to show that this is the best possible. For any antichain F and $x \in F$, the maximal chains passing through x cannot pass through any other element of F. Therefore, if we sum up for each element $x \in F$ the maximal chains passing through it, it cannot be more than total number of maximal chains. For $\mathbf{2}^n$, we then have $\sum_{x \in F} |x|!(n-|x|)! \leqslant n!$. Dividing both sides by $n!$ we get (for $\mathbf{2}^n$ only) $\sum_{x \in F} N_{r(x)}^{-1} \leqslant 1$. Hence $|F| \leqslant \max_x N_{r(x)}$. ■

Definition 12.2.14 The inequality $\sum_{x \in F} N_{r(x)}^{-1} \leqslant 1$ is the ***LYM inequality***. A graded poset satisfies the ***LYM property*** if its antichains all satisfy the LYM inequality. Such a poset is an ***LYM order***.

Remark. By 12.2.6, LYM property implies Sperner property. In fact, LYM property implies strong Sperner property.

Definition 12.2.15 (12.2.18) A nonempty list of maximal chains (not necessarily disjoint) in a graded poset P is a ***regular covering*** of P if, for each rank P_k, each element of P_k lies in the same fraction of these chains.

Definition 12.2.16 (12.2.20) A graded poset P has the ***normalized matching property*** if $|A^*|/N_{k+1} \geqslant |A|/N_k$ for all k and all $A \subseteq P_k$, where $A^* = U[A] \cap P_{k+1}$ ($U[A]$ is the up-set of A).

Theorem 12.2.17 (12.2.22) For a graded poset P, the following statements are equivalent:

- P has a regular covering.
- P has the LYM property.

- P has the normalized matching property.

Theorem 12.2.18 (12.2.24) Every rank-unimodal rank-symmetric LYM poset is a symmetric chain order.

Remark. Chain product $\Rightarrow$ symmetric chain order $\Rightarrow$ strong Sperner property $\Rightarrow$ for all k, k largest ranks form a maximum k-family.

Chapter 14

The probabilistic Method

11/17/2008

Remark. This chapter and anything following it focus on ***methods***. It seems that these methods are complex but not too hard to understand. The important thing to grasp is then to see how methods are being used.

14.1 Existence and Expectation

Topic: Probability Spaces and Inequalities

Proposition 14.1.1 (14.1.5) For $x \in \mathbb{R}$, $1 + x \leqslant e^x$, with equality only for $x = 0$. Also, $(1 + \frac{x}{n})^n < e^x$ for $n \in \mathbb{N}$.

Proposition 14.1.2 (14.1.6) if $k \in \mathbb{N}$, then $\binom{n}{k} < (ne/k)^k$.

Topic: Existence Arguments

Theorem 14.1.3 (14.1.7) $R(k, k) < (e\sqrt{2})^{-1} k 2^{k/2}$.

Proof. The idea is to treat the edges as having $\frac{1}{2}$ probability of being in a graph. There are $\binom{n}{2}$ edges, so each graph has probability of $2^{-\binom{n}{2}}$ to show up. For any k vertices, the probability that it is a clique or independent set is $2 \cdot 2^{-\binom{k}{2}} = 2^{1-\binom{k}{2}}$. There are $\binom{n}{k}$ ways of choosing k vertices, therefore the probability that at least one of them is a k-clique or independent set of size k is at most $\binom{n}{k} 2^{1-\binom{k}{2}}$. If some n makes value is less than 1 then $R(k, k) > n$. ■

Topic: Random Variables

Theorem 14.1.4 (14.1.16) For a graph G, $\alpha(G) \geqslant \sum_{v \in V(G)} \frac{1}{d(v)+1}$.

Proof. Use an ordering σ of vertices of G, if a vertex v appears in the ordering earlier than all its neighbors, we may include it in our independent set. When ordering is uniformly chosen at random, the probability that v appears before all its neighbors is $\frac{1}{d(v)+1}$. Rest follows. ∎

Theorem 14.1.5 (*14.1.17) Let G be an n-vertex graph with no $(r+1)$-clique. The number of edges of G is maximized (uniquely) when G is the complete r-partite graph $T_{n,r}$ whose part-sizes differ by at most 1.

Theorem 14.1.6 (14.1.20) The optimal pebbling number of the k-dimensional hypercube is at least $(4/3)^k$.

11/19/2008

14.2 Refinements of Basic Methods

Topic: Deletions and Alterations

Theorem 14.2.1 (14.2.1) $R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}$, where $n \in \mathbb{N}$. In particular, $R(k,k) > (1-o(1))e^{-1}k2^{k/2}$.

Theorem 14.2.2 (14.2.2) For $k > 1$, every n-vertex graph with minimum degree k has a dominating set of size at most $n\frac{1+\ln(k+1)}{k+1}$.

Theorem 14.2.3 (14.2.3) Given $k \geqslant 3, g \geqslant 3$, there exists a graph with girth at least g and chromatic number at least k.

11/21/2008

Topic: The Symmetric Local Lemma

Definition 14.2.4 Let $A_1, \ldots, A_n$ be events. A ***compound event*** specifies the occurrence of A_i for $i \in S$ and the non-occurrence of A_j for $j \in T$, where S and T are disjoint subsets of $[n]$. An event B is ***mutually independent*** of $A_1, \ldots, A_n$ if B is independent of each compound event specified by disjoint subsets of $[n]$.

Corollary 14.2.5 (Symmetric Local Lemma) Let $A_1, \ldots, A_n$ be events such that each is mutually independent of some set of all but $d-1$ of the other events, and suppose that $P(A_i) \leqslant p$ for all i. If $epd < 1$, then $P(\cap \bar{A}_i) > 0$.

Remark. This part seems to be quite complicated; due to time constraints, it will not be pursued further, at least for now.