Round Compression for Parallel Matching Algorithms

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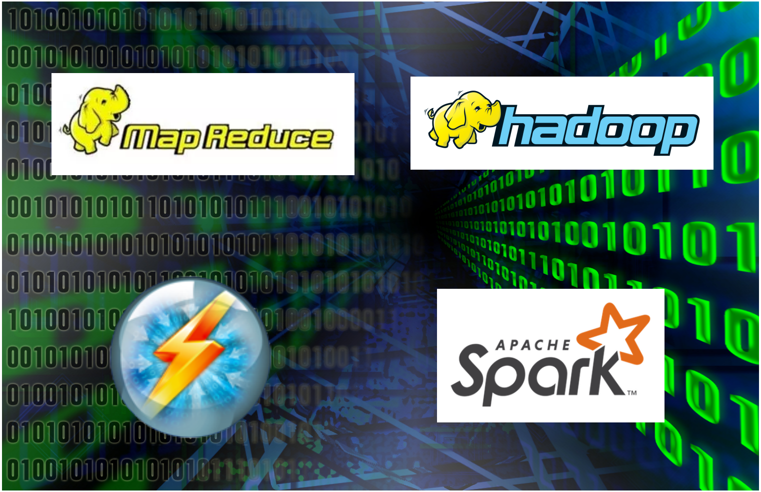
Joint work with **Artur Czumaj** (U of Warwick),
Jakub Łącki (Google), **Aleksander Mądry** (MIT),
Slobodan Mitrović (EPFL), and **Piotr Sankowski** (U of Warsaw)

Mandatory "Big Data" Slides



Massive Data Systems

Outline

1. Model of Computation
2. Graph Matchings in MPC
3. Review of Distributed Algorithms
4. Our Algorithm
5. Further Research

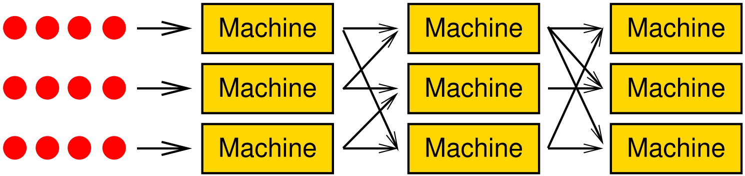
Outline

Model: Massive Parallel Computation (MPC)

Introduced by Karloff, Suri, and Vassilvitskii (2010)

M machines S space per machine

Input: N items



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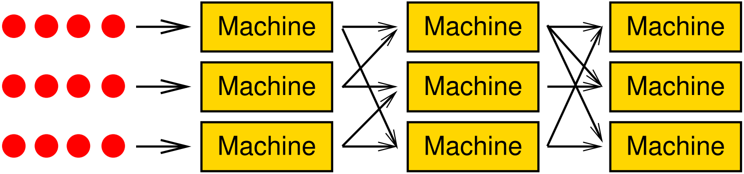

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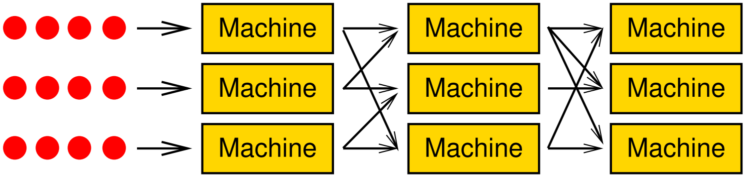

- Initially: each machine receives N/M items
- Single round:
  1. Each machine performs computation
  2. Each machine sends and receives at most $O(S)$ data

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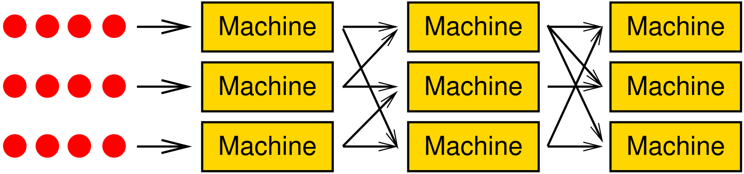
- Goals:
  - Small number of rounds
  - Small space per machine
  - Fast local computation

This Talk: MPC for Graphs

Input: edges of an m-edge graph on n vertices

S space per machine $\qquad$ $M = O(m/S)$ machines



This Talk: Matching Algorithms

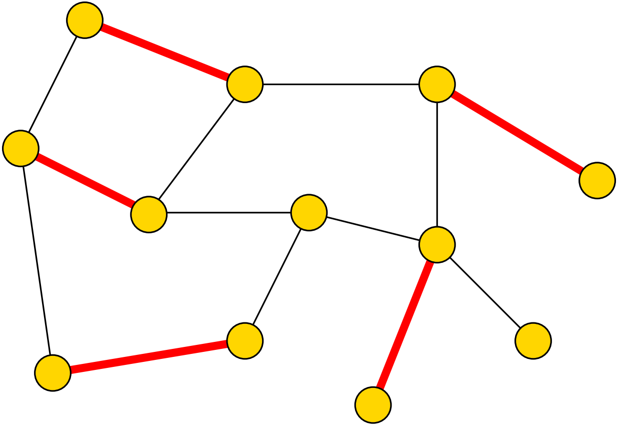
Why study graph matchings?

- Non-trivial appealing packing problem
- Great testbed for many new algorithmic ideas
- Helpful to understand the power of the model
- They have practical applications

Outline

Graph Problems

Maximum Matching: find maximum set of vertex disjoint edges



Large Matchings: Rounds vs. Space

- Space $n^{1+\Omega(1)}$: $(1+\epsilon)$-approximation in $O(1)$ rounds
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- For $O(n)$ space, round complexity becomes $\Theta(\log n)$

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Interesting space regime:

- often just enough to fit a solution on a single machine
- gold standard for space in semi-streaming algorithms
- reasonable middle ground

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Vertex sampling:
Previous algorithms used edge sampling

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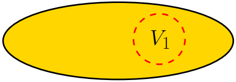

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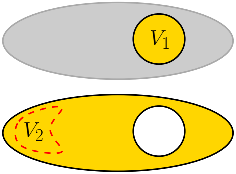

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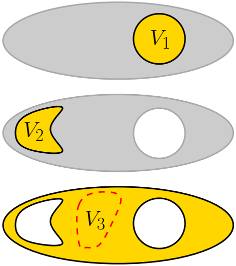

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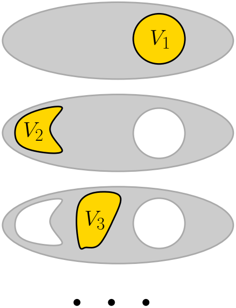

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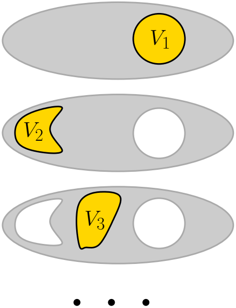

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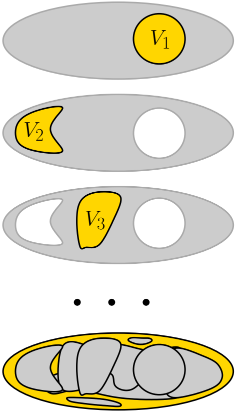
- $\mathcal{C} := \bigcup V_i$ is a vertex cover of size $O(\log n) \cdot \mathrm{OPT}$



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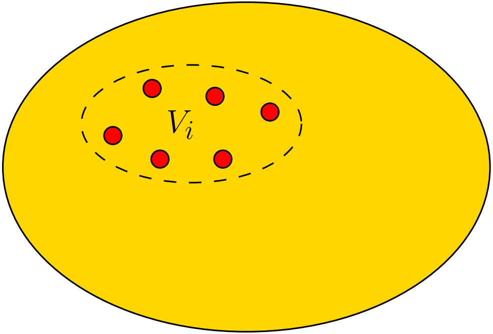

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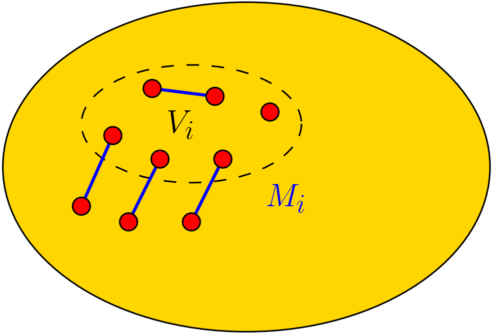

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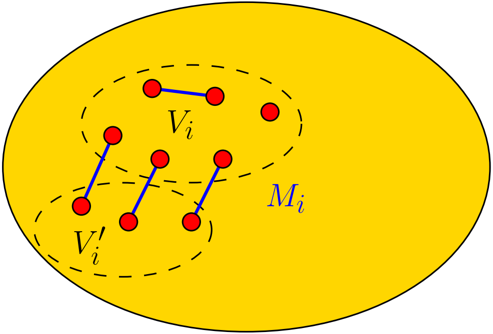

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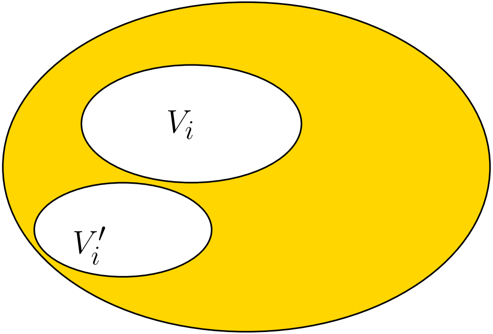

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- Goal: efficiently emulate this algorithm in MPC

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- Our plan:
  - Partition vertices at random into a number of groups
  - Ensure that graphs induced by each group fit onto a single machine
  - Ensure that enough neighbors on the machine to satisfy the properties above

Random Vertex Partitioning

- Phase 1:
  - Partition vertices at random into $\sqrt{n}$ groups
  - Each group should have $O((\sqrt{n})^2) = O(n)$ edges
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- Can do this for roughly $\log(\sqrt{n}) = \frac{1}{2} \log n$ phases:
  - Stuck when max degree gets roughly $\sqrt{n}$
  - Why? Current high degree vertices see no neighbors

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  - Then $\frac{1}{8}\log n$ phases, $\frac{1}{16}\log n$ phases, ...
  - After $O(\log\log n)$ MPC rounds, we would be done

Actual Solution

- We do not know how to analyze this approach directly
- We tweak the peeling algorithm
- Show independence and near-uniformity of surviving vertices
- We get $O\left((\log \log n)^2\right)$ rounds

Outline

Recent Follow-Up Work

Assadi (arXiv 2017)

- Round compression for the Parnas-Ron algorithm
- $O(\log n)$-approximation to vertex cover in $O(\log \log n)$ MPC round
- Bounding technique of Assadi and Khanna (2017)

Recent Follow-Up Work

Assadi (arXiv 2017)

Assadi, Bateni, Bernstein, Mirrokni, and Stein (arXiv 2017)

- Improve round complexity to $O(\log \log n)$
- Approximation improved to $1 + \epsilon$ [McGregor 2005]
- $(2 + \epsilon)$-approximation for vertex cover
- No round compression, but still vertex sampling
- Apply techniques developed for dynamic matching [Bernstein, Stein 2015]

Recent Follow-Up Work

Assadi (arXiv 2017)

Assadi, Bateni, Bernstein, Mirrokni, and Stein (arXiv 2017)

Ghaffari, Gouleakis, Konrad, Mitrović, Rubinfeld (PODC 2018)

- Improve round complexity to $O(\log \log n)$
- Simulate a parallel fractional algorithm
- Explore connections to congested clique model
- Also $O(\log \log n)$-round algorithm for Maximal Independent Set

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    - Limitations of natural sampling techniques?

- Show that a simple very greedy algorithm just works?

Questions?

Full version: `https://arxiv.org/abs/1707.03478`