Theoretically-Efficient and Practical Parallel In-Place Radix Sorting

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Agenda

● Introduction
  ○ Motivation
  ○ Related Work

● Regions Sort: a new relaxed PIP algorithm for radix sort
  ○ Algorithm Design
  ○ Theoretical Analysis

● Experiments
  ○ Setup
  ○ Results
Motivation
Why Radix Sort?

Takes \( O(n) \) work for fixed length integers.

Comparison-based sorts take \( \Omega(n \log(n)) \) work.
In-Place Algorithms

What are in-place algorithms?

● Require at most sublinear auxiliary space.

Why in-place?

● Smaller memory footprint!
● Potentially better utilization of cache.
Radix Sort

- Sort elements according to one digit at a time.
- Most significant digit to least significant digit.
- Recurse on elements with equal digits.
Country: sub-array that will include elements belonging to the same bucket after sorting.
Sort elements according to digits such that each element is in the **correct** country.

**Input:** 0 0 2 0 3 2 1 1 1 3 1

**Output:** 0 0 0 1 1 1 1 2 2 3 3
Serial In-place Radix Sort

1. Find start location of each country (Histogram Building).

2. Move items to the correct country in-place.
Histogram Building

Input: 0 0 2 0 3 2 1 1 1 3 1

Sizes: 3 4 2 2

Prefix sum: 0 3 7 9

Output: 0 0 2 0 3 2 1 1 1 3 1
Parallel Histogram Building

```plaintext
0 0 2 0 3 2 1 1 1 3 1

P_1

P_2

P_3

0 2
1 1
1 1
1 1
3 2
4 4
2 2
```
Serial In-place Radix Sort

Initialize pointer to beginning of each country

For each country:

   While (pointer not at end of country) {
       While (item pointed to is not in correct country) {
           Swap item to location pointed to in target country
           Increment target country pointer
       }
       Increment current country pointer
   }
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   Increment current country pointer
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Why parallel in-place is hard?!
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PARADIS [Cho et. al 2015]
- Parallel in-place radix sort.
- Worst case span is $O(n)$.

IPS4o [Axtmann et. al 2017]
- Parallel in-place comparison based sort.
- Work is $O(n \log(n))$. 
Goal

A parallel in-place algorithm for radix sort

For some parameter $K$:

a. Work: $O(n)$
b. Span: $O(\log(K) + n/K)$
c. Space: $O(K)$

(assuming fixed length integers)
Our Algorithm: Regions Sort
1. Local Sorting
   ○ Partially sort the input.

2. Regions Graph Building
   ○ Represent dependences in partially sorted array with small amount of memory.

3. Global Sorting
   ○ Use regions graph to completely sort the input.
Key Idea:
Divide array into K Blocks and sort each block independently.

Block: sub-array of size n/K.
Local Sorting

Sort using serial in-place radix sort
Key Idea: Represent dependences in partially sorted array with small amount of memory.
Homogeneous sub-array: A subarray with the same digit.
Region: A homogeneous sub-array within same country.
Create edge of weight $W$ from country $x$ to country $y$ if a region of $W$ elements wants to go from country $x$ to country $y$.
Regions Graph Building

No self-edges
Global Sorting

Key Idea: Use regions graph to move regions to their target countries iteratively and updating the graph.

Two Approaches:

1. Cycle Finding
2. 2-Path Finding
A 2-path consists of two edges:

- Incoming edge to node $x$ corresponding to a region that can be moved into country $x$.
- Outgoing edge from node $x$ corresponding to a region that is in country $x$ and needs to be moved out of country $x$. 
Global Sorting: 2-Path Finding

2-path Finding
2-path Finding

1. Choose a vertex.
2-path Finding

1. Choose a vertex.
2. Match incoming edges with outgoing edges.
2-path Finding

1. Choose a vertex.
2. Match incoming edges with outgoing edges.
Global Sorting: 2-Path Finding

2-path Finding

1. Choose a vertex.
2. Match incoming edges with outgoing edges.
3. Execute swaps.
2-path Finding

1. Choose a vertex.
2. Match incoming edges with outgoing edges.
3. Execute swaps.
4. Edit edges.
2-path Finding

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2-path Finding

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Analysis
Analysis

1. Local Sorting
   a. Work: $O(n)$
   b. Span: $O(\log(K) + n/K)$
   c. Space = $O(KB)$

- $K$ is number of blocks
- $B$ is number of buckets per block
Analysis

2. Build Regions Graph

a. Work = \(O(KB)\)

b. Span = \(O(\log(KB))\)

c. Space = \(O(KB)\)

- Since \(#edges \leq #regions \leq KB\)
- \(K\) is number of blocks
- \(B\) is number of buckets per block
3. Global Sorting

a. Work = $O(n)$

b. Span = $O(B (\log(KB) + B))$

c. Space = $O(KB)$

- $O(n)$ swaps
- #nodes removed = $O(B)$
- #edges at each node removed is $O(KB)$
Total for one level of recursion

Work = \( O(n) \)
Span = \( O(n/K + B \log(KB) + B) \)
Space = \( O(KB) \)
Recursion
Recursion

- Each country is recursed on independently.
- Each country divided into number of blocks proportional to its size.
- Integers with range $r$ need at most $\log_B(r)$ recursion levels to be fully sorted.
- For problem sizes smaller than $B$, we use comparison sort.
Algorithm: Recursion

Total on all levels

a. Work = $O(n \log(r))$
b. Span = $O((\log(K) + n/K) \log(r))$
c. Space = $O(P \log(r) + K)$

- Assuming $B = \Theta(1)$
Algorithm: Recursion

Total on all levels

a. Work = \( O(n) \)
b. Span = \( O((\log(K) + n/K)) \)
c. Space = \( O(P + K) \)

- Assuming \( B = \Theta(1) \)
- Assuming \( r = \Theta(1) \) (fixed length integers)
Alternative Approach: Cycle Finding

- Find Cycle in Regions Graph
- Execute Cycle to move elements
- Remove edge with min weight, and decrease weight of all other edges by this weight
- Repeat until all edges are deleted
Evaluation
Evaluation: Control Algorithms

State of the art parallel sorting algorithms:

- `__gnu_parallel::sort` (MCSTL, included in gcc) [Singler et. al 2007]
  - Not fully in-place; uses parallel mergesort
- RADULS (parallel out-of-place radix sort) [Kokot et al. 2017]
- PBBS parallel out-of-place radix sort [Shun et. al 2012]
- PBBS parallel out-of-place sample sort [Shun et. al 2012]
- Ska Sort (serial in-place radix sort)
- IPS4o (parallel in-place sample sort) [Axtmann et al. 2017]
- PARADIS (parallel in-place radix sort) not publicly available

Input distribution:

- Uniform
- Skewed
- Equal, and almost sorted
Evaluation: Our Algorithms

Our Algorithms

Cycle finding
K = P
B = 256

2-path finding
K = 5000
B = 256
Evaluation: Test Environment

- AWS c5.9xlarge
- Intel Xeon Platinum 8000 series
- 72 vCPU (36 cores with hyperthreading)
- 144 GB RAM
- All code compiled with g++-7 with Cilk Plus
Comparison with other algorithms

Regions Sort performance on various inputs with 1 billion integers:

- Between 1.1-3.6x faster than IPS4o, the fastest parallel sample sort, except on one input (1.02x slower).
- Between 1.2-4.4x faster than the fastest out-of-place Radix Sort (PBBS).
- 1.3x slower to 9.4x faster than RADULS.
- About 2x faster than PARADIS based on their reported numbers on same number of cores
Speedup over serial 2-path: 1 billion random integers
Distribution independence: 1 billion integers from Zipf
Regions Sort: fastest across all input sizes (Random)
Input Range - Uniform Sequence (1 billion integers)
Conclusion

Our contributions:

- Regions Sort: the first parallel in-place radix sort with strong theoretical guarantees.

- Empirical evidence showing high scalability and distribution independence.

- Almost always faster than state-of-the-art parallel sorting algorithms in our experiments.