Multicore Triangle Computations Without Tuning

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Presentation is based on paper published in International Conference on Data Engineering (ICDE), 2015
Parallel Cache-Oblivious Sorting

- 32 cores with hyper-threading
- Cache-oblivious sample sort gets near linear speedup and outperforms stlParallelSort by 1.2 to 2.4x
Triangle Computations

• Triangle Counting
  Count = 3

• Other variants:
  • Triangle listing
  • Local triangle counting/clustering coefficients
  • Triangle enumeration
  • Approximate counting
  • Analogs on directed graphs

• Numerous applications…
  • Social network analysis, Web structure, spam detection, outlier detection, dense subgraph mining, 3-way database joins, etc.

Need fast triangle computation algorithms!
Sequential Triangle Computation Algorithms

• Sequential algorithms for exact counting/listing
  • Naïve algorithm of trying all triplets
    $O(V^3)$ work
  • Node-iterator algorithm [Schank]
    $O(VE)$ work
  • Edge-iterator algorithm [Itai-Rodeh]
    $O(VE)$ work
  • Tree-lister [Itai-Rodeh], forward/compact-forward [Schank-Wagner, Lapaty]
    $O(E^{1.5})$ work

• Sequential algorithms via matrix multiplication
  • $O(V^{2.37})$ work compute $A^3$, where $A$ is the adjacency matrix
  • $O(E^{1.41})$ work [Alon-Yuster-Zwick]
  • These require superlinear space
Sequential Triangle Computation Algorithms

Source: “Algorithmic Aspects of Triangle-Based Network Analysis”, Dissertation by Thomas Schank

What about parallel algorithms?
Parallel Triangle Computation Algorithms

• Most designed for distributed memory
  • MapReduce algorithms [Cohen ’09, Suri-Vassilvitskii ‘11, Park-Chung ‘13, Park et al. ‘14]
  • MPI algorithms [Arifuzzaman et al. ‘13, Graphlab]

• What about shared-memory multicore?
  • Multicores are everywhere!
  • Node-iterator algorithm [Green et al. ‘14]
    • $O(VE)$ work in worst case

• Can we obtain an $O(E^{1.5})$ work shared-memory multicore algorithm?
Triangle Computation: Challenges for Shared Memory Machines

1. Irregular computation

2. Deep memory hierarchy
Cache Complexity Model

Complexity = \# \text{cache misses} \text{ disk accesses}

Cache-aware (external-memory) algorithms: have knowledge of M and B
Cache-oblivious algorithms: no knowledge of parameters
Cache Oblivious Model [Frigo et al. ‘99]

- Algorithm works well regardless of cache parameters
- Works well on multi-level hierarchies
- Parallel Cache Oblivious Model for hierarchies of shared and private caches [Blelloch et al. ‘11]

<table>
<thead>
<tr>
<th>Primitive</th>
<th>Work</th>
<th>Depth</th>
<th>Cache Complexity</th>
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<tbody>
<tr>
<td>Scan/filter/merge</td>
<td>$O(n)$</td>
<td>$O(\log n)$</td>
<td>$O(n/B)$</td>
</tr>
<tr>
<td>Sort</td>
<td>$O(n \log n)$</td>
<td>$O(\log^2 n)$</td>
<td>$O((n/B)\log_{(M/B)}(n/B))$</td>
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External-Memory and Cache-Oblivious Triangle Computation

- All previous algorithms are sequential
- External-memory (cache-aware) algorithms
  - Natural-join \( O(E^3/(M^2 B)) \) I/O’s
  - Node-iterator [Dementiev ’06] \( O((E^{1.5}/B) \log_{M/B}(E/B)) \) I/O’s
  - Compact-forward [Menegola ‘10] \( O(E + E^{1.5}/B) \) I/O’s
  - [Chu-Cheng ’11, Hu et al. ‘13] \( O(E^2/(MB) + \#\text{triangles}/B) \) I/O’s
- External-memory and cache-oblivious
  - [Pagh-Silvestri ‘14] \( O(E^{1.5}/(M^{0.5} B)) \) I/O’s or cache misses
- Parallel cache-oblivious algorithms?
Our Contributions

1. Parallel Cache-Oblivious Triangle Counting Algs

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$V = \# \text{ vertices}$  
$E = \# \text{ edges}$  
$M = \text{cache size}$  
$\alpha = \text{arboricity (at most } E^{0.5})$  
$B = \text{line size}$  
$\text{sort}(n) = (n/B) \log_{M/B}(n/B)$

2. Extensions to Other Triangle Computations:
    Enumeration, Listing, Local Counting/Clustering Coefficients, Approx. Counting, Variants on Directed Graphs

3. Extensive Experimental Study
Sequential Triangle Counting (Exact)

(Forward/compact-forward algorithm)

Rank vertices by degree (sorting)
Return $A[v]$ for all $v$ storing higher ranked neighbors

for each vertex $v$:
  for each $w$ in $A[v]$:
    count += intersect($A[v], A[w]$)

Gives all triangles $(v, w, x)$ where $\text{rank}(v) < \text{rank}(w) < \text{rank}(x)$

Work = $O(E^{1.5})$

[Schank-Wagner ‘05, Latapy ‘08]
Proof of $O(E^{1.5})$ work bound when intersect uses merging

1. Rank vertices by degree (sorting)
   Return $A[v]$ for all $v$ storing higher ranked neighbors

2. for each vertex $v$:
   for each $w$ in $A[v]$: 
   
   count += intersect($A[v], A[w]$)

- Step 1: $O(E+V \log V)$ work
- Step 2:
  - For each edge $(v,w)$, intersect does $O(d^+(v) + d^+(w))$ work
  - For all $v$, $d^+(v) \leq E^{0.5}$
    - If $d^+(v) > E^{0.5}$, each of its higher degree neighbors also have degree $> E^{0.5}$ and total number of directed edges $> E$, a contradiction
  - Total work = $E \times O(E^{0.5}) = O(E^{1.5})$
Parallel Triangle Counting (Exact)

**Step 1**
- Work = \(O(E + V \log V)\)
- Depth = \(O(\log^2 V)\)
- Cache = \(O(E + \text{sort}(V))\)

Parallel sort and filter

Rank vertices by degree (sorting)

Return \(A[v]\) for all \(v\) storing higher ranked neighbors

**Parallel for each vertex** \(v\):  
**Parallel for each** \(w\) in \(A[v]\):

\[
\text{count} += \text{intersect}(A[v], A[w])
\]

Parallel reduction

Parallel merge (TC-Merge)

Parallel hash table (TC-Hash)
TC-Merge and TC-Hash Details

**Parallel reduction**

**Step 2: TC-Merge**
- Work = $O(E^{1.5})$
- Depth = $O(\log^2 E)$
- Cache = $O(E + E^{1.5}/B)$

**Step 2: TC-Hash**
- Work = $O(\alpha E)$
- Depth = $O(\log E)$
- Cache = $O(\alpha E)$

($\alpha =$ arboricity (at most $E^{0.5}$))

- **TC-Merge**
  - Preprocessing: sort adjacency lists

- **TC-Hash**
  - Preprocessing: for each vertex, create parallel hash table storing edges [Shun-Blelloch ‘14]
  - Intersect: scan smaller list, querying hash table of larger list in parallel
### Comparison of Complexity Bounds

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<td>$O(\log E)$</td>
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$M = \text{cache size}        B = \text{line size}         \text{sort}(n) = (n/B) \log_{M/B}(n/B)$
Our Contributions

1. **Parallel Cache-Oblivious Triangle Counting Algs**

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V = # vertices  
E = # edges  
M = cache size  
B = line size  
α = arboricity (at most E^{0.5})  
sort(n) = (n/B) log_{M/B}(n/B)

2. **Extensions to Other Triangle Computations:**
   - Enumeration, Listing, Local Counting/Clustering Coefficients
   - Approx. Counting, Variants on Directed Graphs

3. **Extensive Experimental Study**
Extensions of Exact Counting Algorithms

- Triangle enumeration
  - Call `emit` function whenever triangle is found
  - **Listing**: add to hash table to list; return contents at the end
  - **Local counting/clustering coefficients**: atomically increment count of three triangle endpoints

- Directed triangle counting/enumeration
  - Keep separate counts for different types of triangles

- Approximate counting
  - Use colorful triangle sampling scheme to create smaller sub-graph [Pagh-Tsourakakis ‘12]
  - Run TC-Merge or TC-Hash on sub-graph with $pE$ edges ($0 < p < 1$) and return $\#\text{triangles}/p^2$ as estimate
Approximate Counting

- Colorful triangle counting [Pagh-Tsourakakis ’12]

  Sampling rate: $0 < p < 1$

1. Assign random color in $\{1, \ldots, 1/p\}$ to each vertex

2. Sampling: Keep edges whose endpoints have the same color

3. Run exact triangle counting on sampled graph, return $\Delta_{\text{sampled}}/p^2$

Steps 1 & 2

- Work = $O(E)$
- Depth = $O(\log E)$
- Cache = $O(E/B)$

Step 3: TC-Merge

- Work = $O((pE)^{1.5})$
- Depth = $O(\log^2 E)$
- Cache = $O(pE+(pE)^{1.5}/B)$

Step 3: TC-Hash

- Work = $O(V \log V + \alpha pE)$
- Depth = $O(\log E)$
- Cache = $O(\text{sort}(V)+p\alpha E)$

Expected # edges = $pE$
Our Contributions

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3. Extensive Experimental Study
Experimental Setup

- Implementations using Intel Cilk Plus
- 40-core Intel Nehalem machine (with 2-way hyper-threading)
  - 4 sockets, each with 30MB shared L3 cache, 256KB private L2 caches
- Sequential TC-Merge as baseline (faster than existing sequential implementations)
- Other multicore implementations: Green et al. and GraphLab
- Our parallel Pagh-Silvestri algorithm was not competitive
- Variety of real-world and artificial graphs
Both TC-Merge and TC-Hash scale well with # of cores:

**LiveJournal**
4M vtxes, 34.6M edges

**Orkut**
3M vtxes, 117M edges
40-core (with hyper-threading) Performance

- TC-Merge always faster than TC-Hash (by 1.3—2.5x)
- TC-Merge always faster than Green et al. or GraphLab (by 2.1—5.2x)
Why is TC-Merge faster than TC-Hash?

- TC-Hash less cache-efficient than TC-Merge
- Running time more correlated with cache misses than work
Comparison to existing counting algs.

Twitter graph (41M vertices, 1.2B undirected edges, 34.8B triangles)

- **Yahoo graph** (1.4B vertices, 6.4B edges, 85.8B triangles) on 40 cores: TC-Merge takes 78 seconds
  - Approximate counting algorithm achieves 99.6% accuracy in 9.1 seconds
Approximate counting

<table>
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<tr>
<th>p = 1/25</th>
<th>Accuracy</th>
<th>$T_{\text{approx}}$</th>
<th>$T_{\text{approx}}/T_{\text{exact}}$</th>
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<tr>
<td>Orkut (V=3M, E=117M)</td>
<td>99.8%</td>
<td>0.067sec</td>
<td>0.035</td>
</tr>
<tr>
<td>Twitter (V=41M, E=1.2B)</td>
<td>99.9%</td>
<td>2.4sec</td>
<td>0.043</td>
</tr>
<tr>
<td>Yahoo (V=1.4B, E=6.4B)</td>
<td>99.6%</td>
<td>9.1sec</td>
<td>0.117</td>
</tr>
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**Conclusion**

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- Simple multicore algorithms for triangle computations are provably work-efficient, low-depth and cache-friendly
- Implementations require no load-balancing or tuning for cache
- Experimentally outperforms existing multicore and distributed algorithms

- Future work: Design a practical parallel algorithm achieving $O(E^{1.5}/(M^{0.5} B))$ cache complexity