Multicore Triangle Computations Without Tuning

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Presentation is based on paper published in International Conference on Data Engineering (ICDE), 2015
Triangle Computations

- **Triangle Counting**
  Count = 3

- **Other variants:**
  - Triangle listing
  - Local triangle counting/clustering coefficients
  - Triangle enumeration
  - Approximate counting
  - Analogs on directed graphs

- **Numerous applications…**
  - Social network analysis, Web structure, spam detection, outlier detection, dense subgraph mining, 3-way database joins, etc.

Need fast triangle computation algorithms!
Sequential Triangle Computation Algorithms

V = # vertices E = # edges

• Sequential algorithms for exact counting/listing
  • Naïve algorithm of trying all triplets
    \(O(V^3)\) work
  • Node-iterator algorithm [Schank]
    \(O(VE)\) work
  • Edge-iterator algorithm [Itai-Rodeh]
    \(O(VE)\) work
  • Tree-lister [Itai-Rodeh], forward/compact-forward [Schank-Wagner, Lapaty]
    \(O(E^{1.5})\) work

• Sequential algorithms via matrix multiplication
  • \(O(V^{2.37})\) work compute \(A^3\), where \(A\) is the adjacency matrix
  • \(O(E^{1.41})\) work [Alon-Yuster-Zwick]
  • These require superlinear space
Sequential Triangle Computation Algorithms

What about parallel algorithms?

Source: “Algorithmic Aspects of Triangle-Based Network Analysis”, Dissertation by Thomas Schank
Parallel Triangle Computation Algorithms

- Most designed for distributed memory
  - MapReduce algorithms [Cohen ’09, Suri-Vassilvitskii ‘11, Park-Chung ‘13, Park et al. ‘14]
  - MPI algorithms [Arifuzzaman et al. ‘13, Graphlab]

- What about shared-memory multicore?
  - Multicores are everywhere!
  - Node-iterator algorithm [Green et al. ‘14]
    - $O(VE)$ work in worst case

- Can we obtain an $O(E^{1.5})$ work shared-memory multicore algorithm?
Triangle Computation: Challenges for Shared Memory Machines

1. Irregular computation

2. Deep memory hierarchy
External-Memory and Cache-Oblivious Triangle Computation

- All previous algorithms are sequential
- External-memory (cache-aware) algorithms
  - Natural-join $O(E^3/(M^2 B))$ I/O’s
  - Node-iterator [Dementiev ’06] $O((E^{1.5}/B) \log_{M/B}(E/B))$ I/O’s
  - Compact-forward [Menegola ‘10] $O(E + E^{1.5}/B)$ I/O’s
  - [Chu-Cheng ’11, Hu et al. ‘13] $O(E^2/(MB) + \#triangles/B)$ I/O’s
- External-memory and cache-oblivious
  - [Pagh-Silvestri ‘14] $O(E^{1.5}/(M^{0.5} B))$ I/O’s or cache misses
- Parallel cache-oblivious algorithms?
Our Contributions

1. **Parallel Cache-Oblivious Triangle Counting Algs**

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<th>Algorithm</th>
<th>Work</th>
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$V = \#$ vertices  \quad E = \#$ edges  \quad \alpha = \text{arboricity (at most } E^{0.5})$

$M = \text{cache size}  \quad B = \text{line size}  \quad \text{sort}(n) = (n/B) \log_{M/B}(n/B)$

2. **Extensions to Other Triangle Computations:**
   - Enumeration, Listing, Local Counting/Clustering Coefficients,
   - Approx. Counting, Variants on Directed Graphs

3. **Extensive Experimental Study**
Sequential Triangle Counting (Exact)

*(Forward/compact-forward algorithm)*

1. Rank vertices by degree (sorting)
   Return $A[v]$ for all $v$ storing higher ranked neighbors

2. for each vertex $v$:
   for each $w$ in $A[v]$:
     count += intersect($A[v], A[w]$)

Gives all triangles $(v, w, x)$ where $\text{rank}(v) < \text{rank}(w) < \text{rank}(x)$

Work = $O(E^{1.5})$

[Schank-Wagner ‘05, Latapy ‘08]
Proof of $O(E^{1.5})$ work bound when intersect uses merging

1. Rank vertices by degree (sorting)
   Return $A[v]$ for all $v$ storing higher ranked neighbors

2. for each vertex $v$:
   for each $w$ in $A[v]$:
   count $+= \text{intersect}(A[v], A[w])$

- **Step 1:** $O(E + V \log V)$ work
- **Step 2:**
  - For each edge $(v, w)$, intersect does $O(d^+(v) + d^+(w))$ work
  - For all $v$, $d^+(v) \leq E^{0.5}$
    - If $d^+(v) > E^{0.5}$, each of its higher degree neighbors also have degree $> E^{0.5}$ and total number of directed edges $> E$, a contradiction
  - Total work $= E \times O(E^{0.5}) = O(E^{1.5})$
Parallel Triangle Counting (Exact)

Step 1
Work = O(E+V log V)
Depth = O(log^2 V)
Cache = O(E+sort(V))

Parallel sort and filter

Rank vertices by degree (sorting)
Return $A[v]$ for all $v$ storing higher ranked neighbors

Parallel reduction

parallel_for each vertex $v$:

parallel_for each $w$ in $A[v]$:

count += intersect($A[v], A[w]$)

parallel_merge (TC-Merge)

Parallel hash table (TC-Hash)
TC-Merge and TC-Hash Details

Parallel reduction

\[
\text{for each vertex } v:\n\text{for each } w \text{ in } A[v]:\n\text{count } += \text{intersect}(A[v], A[w])
\]

Step 2: TC-Merge
\[
\begin{align*}
\text{Work} &= O(E^{1.5}) \\
\text{Depth} &= O(\log^2 E) \\
\text{Cache} &= O(E + E^{1.5}/B)
\end{align*}
\]

- TC-Merge
  - Preprocessing: sort adjacency lists

Step 2: TC-Hash
\[
\begin{align*}
\text{Work} &= O(\alpha E) \\
\text{Depth} &= O(\log E) \\
\text{Cache} &= O(\alpha E)
\end{align*}
\]

(\(\alpha = \text{arboricity (at most } E^{0.5}\))

- TC-Hash
  - Preprocessing: for each vertex, create parallel hash table storing edges [Shun-Blelloch ‘14]
  - Intersect: scan smaller list, querying hash table of larger list in parallel

Parallel merge (TC-Merge) or Parallel hash table (TC-Hash)
## Comparison of Complexity Bounds

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<td>$O(VE)$</td>
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3. **Extensive Experimental Study**
Extensions of Exact Counting Algorithms

• Triangle enumeration
  • Call **emit** function whenever triangle is found
  • **Listing**: add to hash table to list; return contents at the end
  • **Local counting/clustering coefficients**: atomically increment count of three triangle endpoints

• Directed triangle counting/enumeration
  • Keep separate counts for different types of triangles

• Approximate counting
  • Use colorful triangle sampling scheme to create smaller sub-graph
    [Pagh-Tsourakakis ‘12]
  • Run TC-Merge or TC-Hash on sub-graph with pE edges (0 < p < 1) and return \#triangles/p^2 as estimate
Approximate Counting

- **Colorful triangle counting** [Pagh-Tsourakakis ’12]
  
  Sampling rate: 0 < p < 1

  1. **Parallel scan**
     - Assign random color in \{1, …, 1/p\} to each vertex

  2. **Parallel filter**
     - Sampling: Keep edges whose endpoints have the same color

  3. **Use TC-Merge or TC-Hash**
     - Run exact triangle counting on sampled graph, return \(\Delta_{\text{sampled}}/p^2\)

**Steps 1 & 2**
- Work = \(O(E)\)
- Depth = \(O(\log E)\)
- Cache = \(O(E/B)\)

**Step 3: TC-Merge**
- Work = \(O((pE)^{1.5})\)
- Depth = \(O(\log^2 E)\)
- Cache = \(O(pE+(pE)^{1.5}/B)\)

**Step 3: TC-Hash**
- Work = \(O(V \log V + \alpha pE)\)
- Depth = \(O(\log E)\)
- Cache = \(O(\text{sort}(V)+p\alpha E)\)

Expected # edges = \(pE\)
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$V = \#$ vertices, $E = \#$ edges, $\alpha = \text{arboricity (at most } E^{0.5})$, $M = \text{cache size}$, $B = \text{line size}$, $\log^m (n/B) = (n/B) \log_{M/B}(n/B)$

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3. **Extensive Experimental Study**
Experimental Setup

- Implementations using Intel Cilk Plus
- 40-core Intel Nehalem machine (with 2-way hyper-threading)
  - 4 sockets, each with 30MB shared L3 cache, 256KB private L2 caches
- Sequential TC-Merge as baseline (faster than existing sequential implementations)
- Other multicore implementations: Green et al. and GraphLab
- Our parallel Pagh-Silvestri algorithm was not competitive
- Variety of real-world and artificial graphs
Both TC-Merge and TC-Hash scale well with # of cores:

**LiveJournal**
4M vtxes, 34.6M edges

**Orkut**
3M vtxes, 117M edges
40-core (with hyper-threading) Performance

- TC-Merge always faster than TC-Hash (by 1.3—2.5x)
- TC-Merge always faster than Green et al. or GraphLab (by 2.1—5.2x)
Why is TC-Merge faster than TC-Hash?

- TC-Hash less cache-efficient than TC-Merge
- Running time more correlated with cache misses than work
Comparison to existing counting algs.

Twitter graph (41M vertices, 1.2B undirected edges, 34.8B triangles)

• Yahoo graph (1.4B vertices, 6.4B edges, 85.8B triangles) on 40 cores: TC-Merge takes 78 seconds
  – Approximate counting algorithm achieves 99.6% accuracy in 9.1 seconds
Shared vs. distributed memory costs

• Amazon EC2 pricing
  • Captures purchasing costs, maintenance/operating costs, energy costs

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<tr>
<th>Triangle Counting (Twitter)</th>
<th>Our algorithm</th>
<th>GraphLab</th>
<th>GraphLab</th>
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<tr>
<td>Running Time</td>
<td>0.932 min</td>
<td>3 min</td>
<td>1.5 min</td>
</tr>
<tr>
<td>Machine</td>
<td>40-core (256 GB memory)</td>
<td>40-core (256 GB memory)</td>
<td>64 x 16-core</td>
</tr>
<tr>
<td>Approx. EC2 pricing</td>
<td>&lt; $4/hour</td>
<td>&lt; $4/hour</td>
<td>64 x $0.928/hour</td>
</tr>
<tr>
<td>Overall cost</td>
<td>&lt; $0.062</td>
<td>&lt; $0.2</td>
<td>$1.49</td>
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Approximate counting

\[
\frac{T_{\text{approx}}}{T_{\text{exact}}}
\]

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<th>p=1/25</th>
<th>Accuracy</th>
<th>(T_{\text{approx}})</th>
<th>(T_{\text{approx}}/T_{\text{exact}})</th>
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<td>Orkut (V=3M, E=117M)</td>
<td>99.8%</td>
<td>0.067sec</td>
<td>0.035</td>
</tr>
<tr>
<td>Twitter (V=41M, E=1.2B)</td>
<td>99.9%</td>
<td>2.4sec</td>
<td>0.043</td>
</tr>
<tr>
<td>Yahoo (V=1.4B, E=6.4B)</td>
<td>99.6%</td>
<td>9.1sec</td>
<td>0.117</td>
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Conclusion

- Simple multicore algorithms for triangle computations are provably work-efficient, low-depth, and cache-efficient
- Implementations require no load-balancing or tuning for cache
- Experimentally outperforms existing multicore and distributed algorithms

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- Future work: Design a practical parallel algorithm achieving $O(E^{1.5}/(M^{0.5} B))$ cache complexity