Theoretically-Efficient and Practical Parallel In-Place Radix Sorting

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Why Radix Sort?

Takes $O(n)$ work for fixed length integers.

Comparison-based sorts take $\Omega(n \log(n))$ work.
Radix Sort

- Sort elements according to one digit at a time.
- Most significant digit to least significant digit.
- Recurse on elements with equal digits.
**Country**: sub-array that will include elements belonging to the same bucket after sorting.

**Input:**

```
0 0 2 0 3 2 1 1 1 3 1
```

**Output:**

```
0 0 0 1 1 1 1 2 2 3 3
```
Radix Sort: Subproblem

Sort elements according to digits such that each element is in the correct country.

Input: 0 0 2 0 3 2 1 1 1 3 1

Output: 0 0 0 1 1 1 1 1 2 2 3 3
1. Find start location of each country (Histogram Building).

2. Move items to the correct country in-place.
Histogram Building

Input: 0 0 2 0 3 2 1 1 1 3 1

Sizes: 3 4 2 2

Prefix sum: 0 3 7 9

Output: 0 0 2 0 3 2 1 1 1 3 1
Parallel Histogram Building

\[
\begin{array}{cccccccc}
0 & 0 & 2 & 0 & 3 & 2 & 1 & 1 & 1 & 3 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
P_1 & & & & & & & \\
0 & 2 & & & & & & \\
P_2 & & & & & & & \\
1 & 1 & 1 & & & & & \\
P_3 & & & & & & & \\
3 & 4 & 2 & 2 & & & & \\
\end{array}
\]
Initialize pointer to beginning of each country

For each country:

While (pointer not at end of country) {

    While (item pointed to is not in correct country) {

        Swap item to location pointed to in target country

        Increment target country pointer

    }

    Increment current country pointer

}
Serial In-place Radix Sort

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Why parallel in-place is hard?!
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Related Work

PARADIS [Cho et. al 2015]
- Parallel in-place radix sort.
- Worst case span is $O(n)$.

IPS4o [Axtmann et. al 2017]
- Parallel in-place comparison based sort.
- Work is $O(n\log(n))$. 
A relaxed PIP algorithm for radix sort

For some parameter K:

a. Work: $O(n)$
b. Span: $O(\log(K) + n/K)$
c. Space: $O(K)$

(assuming fixed length integers)
Our Algorithm: Regions Sort
1. Local Sorting
   ○ Partially sort the input.

2. Regions Graph Building
   ○ Represent dependencies in partially sorted array with small amount of memory.

3. Global Sorting
   ○ Use regions graph to completely sort the input.
Local Sorting

Key Idea:

Divide array into K Blocks and sort each block independently.

Block: sub-array of size n/K.
Local Sorting

Sort using serial in-place radix sort
Key Idea: Represent dependences in partially sorted array with small amount of memory.
Homogeneous sub-array: A subarray with the same digit
Regions Graph Building

Region: A homogeneous sub-array within same current country.
Create edge of weight $W$ from country $x$ to country $y$ if a region of $W$ elements wants to go from country $x$ to country $y$. 
Regions Graph Building

No self-edges
Key Idea: Use regions graph to move regions to their target countries iteratively and updating the graph.

Two Approaches:
1. Cycle Finding
2. 2-Path Finding
A 2-path consists of two edges:

- Incoming edge to node $x$ corresponding to a region that can be moved into country $x$.

- Outgoing edge from node $x$ corresponding to a region that is in country $x$ and needs to be moved out of country $x$. 
Global Sorting: 2-Path Finding

2-path Finding

0 0 2 0 1 2 3 1 1 1 1 3

Diagram:

- Nodes: 0, 1, 2, 3
- Edges: 0→1, 1→0, 1→2, 2→1, 1→3, 3→1, 1→0, 0→1, 2→3, 3→2
- Weights: 1 for all edges

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Global Sorting: 2-Path Finding

2-path Finding

1. Choose a vertex.
Global Sorting: 2-Path Finding

2-path Finding

1. Choose a vertex.
2. Match incoming edges with outgoing edges.
Global Sorting: 2-Path Finding

2-path Finding

1. Choose a vertex.
2. Match incoming edges with outgoing edges.
2-path Finding

1. Choose a vertex.
2. Match incoming edges with outgoing edges.
3. Execute swaps.
2-path Finding

1. Choose a vertex.
2. Match incoming edges with outgoing edges.
3. Execute swaps.
4. Edit edges.
Global Sorting: 2-Path Finding

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Analysis
Analysis

1. Local Sorting
   
   a. Work: $O(n)$
   
   b. Span: $O(\log(K) + n/K)$
   
   c. Space = $O(KB)$

- $K$ is number of blocks
- $B$ is number of buckets per block
2. Build Regions Graph

a. Work = $O(KB)$
b. Span = $O(\log(KB))$
c. Space = $O(KB)$

- Since $\#\text{edges} \leq \#\text{regions} \leq KB$
- $K$ is number of blocks
- $B$ is number of buckets per block
3. Global Sorting

- Work = $O(n)$
- Span = $O(B \cdot (\log(KB) + B))$
- Space = $O(KB)$

- $O(n)$ swaps
- #nodes removed = $O(B)$
- #edges at each node removed is $O(KB)$
Total for one level of recursion

Work = $O(n)$
Span = $O(n/K + B \ (\log(KB) + B))$
Space = $O(KB)$
Recursion
Recursion

- Each country is recursed on independently.
- Each country divided into number of blocks proportional to its size.
- Integers with range r need at most $\log_B(r)$ recursion levels to be fully sorted.
- For problem sizes smaller than B, we use comparison sort.
Algorithm: Recursion

Total on all levels

a. Work = $O(n \log(r))$
b. Span = $O((\log(K) + n/K) \log(r))$
c. Space = $O(P \log(r) + K)$

- Assuming $B = \Theta(1)$
Algorithm: Recursion

Total on all levels

a. Work = $O(n)$
b. Span = $O((\log(K) + n/K))$
c. Space = $O(P + K)$

- Assuming $B = \Theta(1)$
- Assuming $r = \Theta(1)$ (fixed length integers)
Alternative Approach: Cycle Finding

- Find Cycle in Regions Graph
- Execute Cycle to move elements
- Remove edge with min weight, and decrease weight of all other edges by this weight
- Repeat until all edges are deleted
Evaluation
Evaluation: Control Algorithms

State of the art parallel sorting algorithms:

- _gnu_parallel::sort (MCSTL, included in gcc) [Singler et. al 2007]
  - Not fully in-place; uses parallel mergesort
- RADULS (parallel out-of-place radix sort) [Kokot et al. 2017]
- PBBS parallel out-of-place radix sort [Shun et. al 2012]
- PBBS parallel out-of-place sample sort [Shun et. al 2012]
- Ska Sort (serial in-place radix sort)
- IPS4o (parallel in-place sample sort) [Axtmann et al. 2017]
- PARADIS (parallel in-place radix sort) not publicly available

Input distribution:

- Uniform
- Skewed
- Equal, and almost sorted
Evaluation: Our Algorithms

Our Algorithms

Cycle finding
  $K = P$
  $B = 256$

2-path finding
  $K = 5000$
  $B = 256$
Evaluation: Test Environment

- AWS c5.9xlarge
- Intel Xeon Platinum 8000 series
- 72 vCPU (36 cores with hyperthreading)
- 144 GB RAM
- All code compiled with g++-7 with Cilk Plus
Regions Sort performance on various inputs with 1 billion integers:

- Between 1.1-3.6x faster than IPS4o, the fastest parallel sample sort, except on one input (1.02x slower).
- Between 1.2-4.4x faster than the fastest out-of-place Radix Sort (PBBS).
- 1.3x slower to 9.4x faster than RADULS.
- About 2x faster than PARADIS based on their reported numbers on same number of cores.
Speedup over serial 2-path: 1 billion random integers
Distribution independence: 1 billion integers from Zipf
Regions Sort: fastest across all input sizes (Random)
Input Range - Uniform Sequence (1 billion integers)
Conclusion

Our contributions:

● Regions Sort: the first parallel in-place radix sort with strong theoretical guarantees.

● Empirical evidence showing high scalability and distribution independence.

● Almost always faster than state-of-the-art parallel sorting algorithms in our experiments.