ADMM for Earth Mover's Distances

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This document derives the ADMM algorithm stated in "Earth Mover's Distances on Discrete Surfaces" (SIGGRAPH 2014).

In §4.5, we reduce the problem of computing earth mover's distances on triangulated surfaces to the following optimization for vector c:

$$\inf_c \sum_t \|B_t c + w_t\|.$$

To derive an ADMM approach, we define per-triangle vectors J_t and solve the following optimization instead:

$$\begin{array}{rl} \inf_{c,J} & \sum_t \|J_t\| \\ \text{s.t.} & J_t = B_t c + w_t \end{array}$$

This optimization problem has the following augmented Lagrangian:

$$L_\beta = \sum_t \left[\|J_t\| + y_t^\top (J_t - B_t c - w_t) + \frac{\beta}{2}\|J_t - B_t c - w_t\|^2 \right]$$

ADMM alternates between three steps detailed below:

$$J \leftarrow \arg\min_J L_\beta(J, c, y)$$
$$c \leftarrow \arg\min_c L_\beta(J, c, y)$$
$$y_t \leftarrow y_t + \beta(J_t - B_t c - w_t)$$

1 J update

We can optimize L_β over J independently for each face since the sum over t decouples in this step. Defining $J_t^0 = B_t c + w_t$ and henceforth in this section dropping the t subscript, we wish to solve

$$\min_J \left[\|J\| + y^\top J + \frac{\beta}{2}\|J - J^0\|^2 \right]$$

This objective is convex, and we could run generic machinery. But in fact we can solve this problem in closed form via the derivation below.

Let's simplify the optimization objective by "completing the square:"

$$\|J\| + y^\top J + \frac{\beta}{2}\|J - J^0\|^2 = \|J\| + y^\top J + \frac{\beta}{2}(\|J\|^2 - 2(J^0)^\top J) + \text{const.}$$

$$
\begin{aligned}
&= \|J\| + \frac{\beta}{2}\|J\|^2 + (y - \beta J^0)^\top J + \text{const.} \\
&= \|J\| + \frac{\beta}{2}\left[\|J\|^2 + \frac{2}{\beta}(y - \beta J^0)^\top J\right] + \text{const.} \\
&= \|J\| + \frac{\beta}{2}\left[\|J\|^2 - 2z^\top J\right] + \text{const.} \\
&= \|J\| + \frac{\beta}{2}\|J - z\|^2 + \text{const.}
\end{aligned}
$$

Here, we defined $z \equiv -\frac{1}{\beta}(y - \beta J^0)$. So, we equivalently can solve the following optimization:

$$\min_J \left[\|J\| + \frac{\beta}{2}\|J - z\|^2\right]$$

In this form, it is clear we can write $J = az$ for some $a \in \mathbb{R}$ (to prove this separate J into components orthogonal and parallel to z; the former must be zero). Then, we can write

$$\min_a \left[|a|\|z\| + \frac{\beta}{2}(a-1)^2\|z\|^2\right]$$

Or, equivalently:

$$\min_a \left[|a| + d(a-1)^2\right],$$

where $d = \frac{\beta}{2}\|z\|$. This final simplification is solvable using elementary techniques. Clearly $a \in [0,1]$, so $|a| = a$. If $f(a) = a + d(a-1)^2$, then $f'(a) = 1 + 2d(a-1) = 0 \implies a = 1 - \frac{1}{2d}$. We have $a > 0 \iff 1 - \frac{1}{2d} > 0 \iff d > 1/2$. Hence, in the end we must have:

$$a = \begin{cases} 1 - \frac{1}{2d} & d > \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

2 c update

For this update step, we can write:

$$0 = \nabla_c L_\beta = \sum_t \left[-B_t^\top y_t - \beta B_t^\top (J_t - w_t) + \beta B_t^\top B_t c\right]$$

Dividing by β and moving terms shows:

$$\left(\sum_t B_t^\top B_t\right) c = \sum_t B_t^\top \left(\frac{y_t}{\beta} + J_t - w_t\right)$$

This is a *small* matrix solve if we use the Laplace-Beltrami basis, and it can be prefactored.

3 Algorithm Summary

Based on the derivation above, the algorithm below minimizes the EMD energy (any time there is a t subscript, there should be a loop over triangles t):

1. J update:

$$z_t \leftarrow B_t c + w_t - \frac{y_t}{\beta}$$
$$a_t \leftarrow \begin{cases} 1 - \frac{1}{\beta\|z_t\|} & \beta\|z_t\| > 1 \\ 0 & \text{otherwise} \end{cases}$$
$$J_t \leftarrow a_t z_t$$

2. c update (can pre-factor the inverted matrix):

$$c \leftarrow \left(\sum_t B_t^\top B_t\right)^{-1} \left[\sum_t B_t^\top \left(\frac{y_t}{\beta} + J_t - w_t\right)\right]$$

3. Dual update:

$$y_t \leftarrow y_t + \beta(J_t - B_t c - w_t)$$