Image by K. Crane

Laplacian Operators

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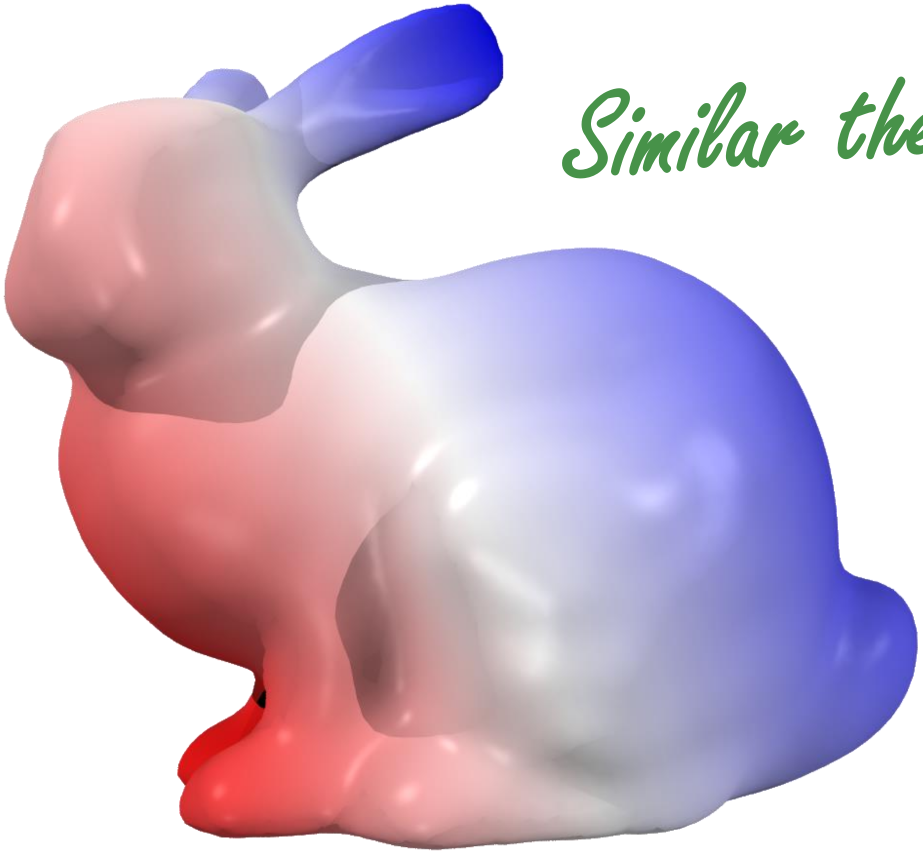
Stanford University

"Laplace-Beltrami: The Swiss Army Knife of Geometry Processing"

SGP 2014 tutorial

J. Solomon, K. Crane, and E. Vouga

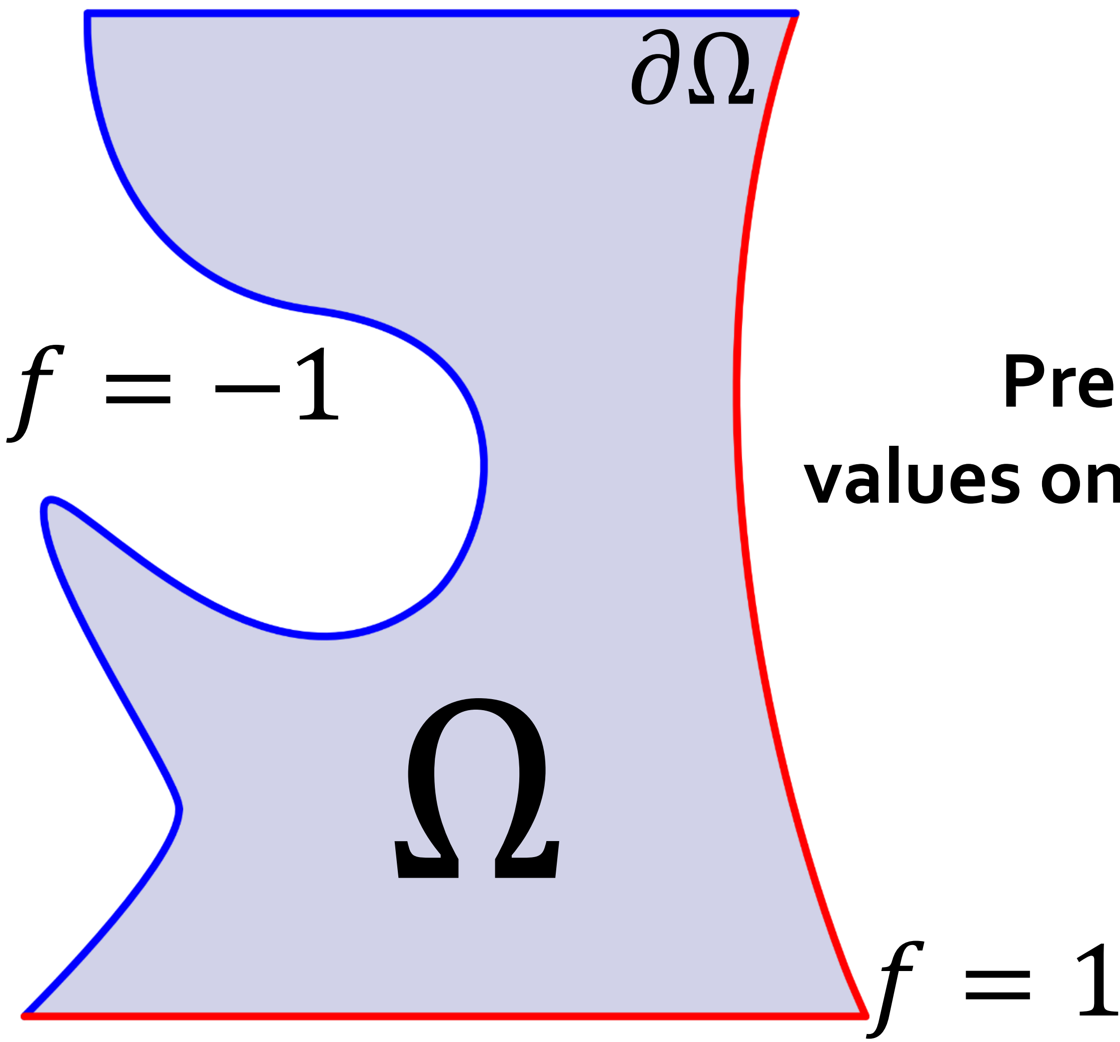
My Background



Similar themes in ML!

Geometry processing

Motivation: Interpolation Problem



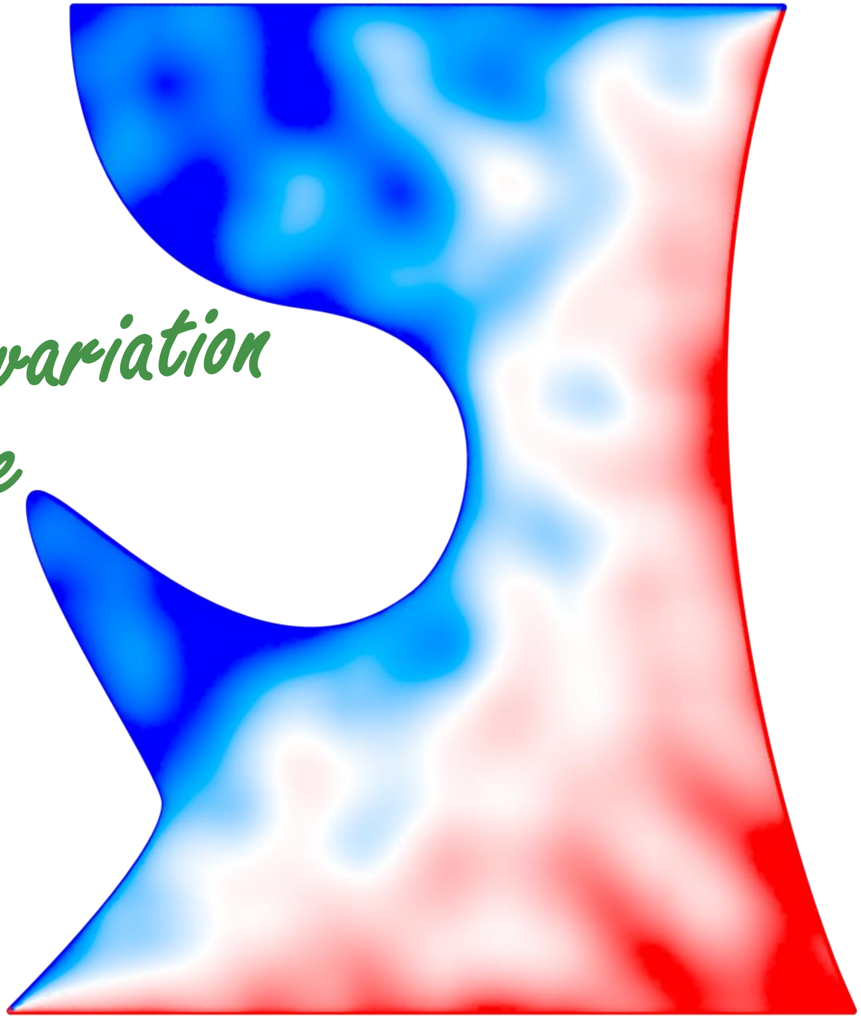
Predict $f: \Omega \to \mathbb{R}$ from values on the boundary $\partial\Omega$.

Desired: Smooth Functions

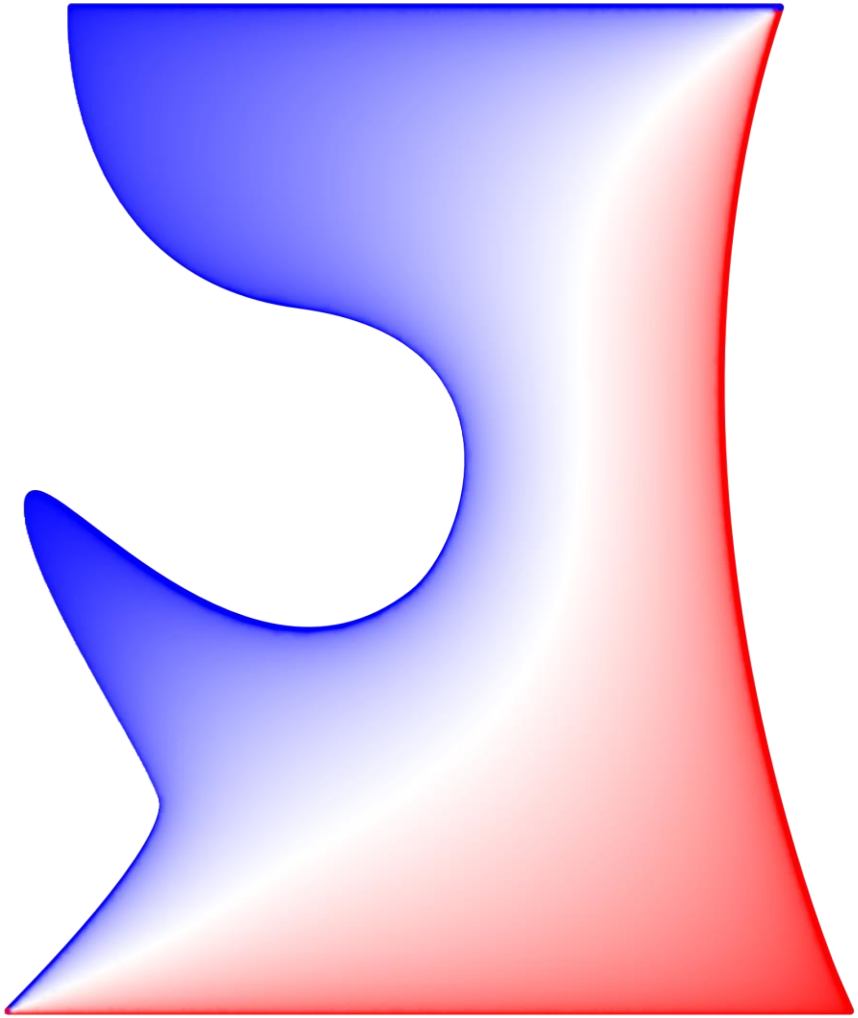
No discontinuities allowed!

Desired: Smooth Functions

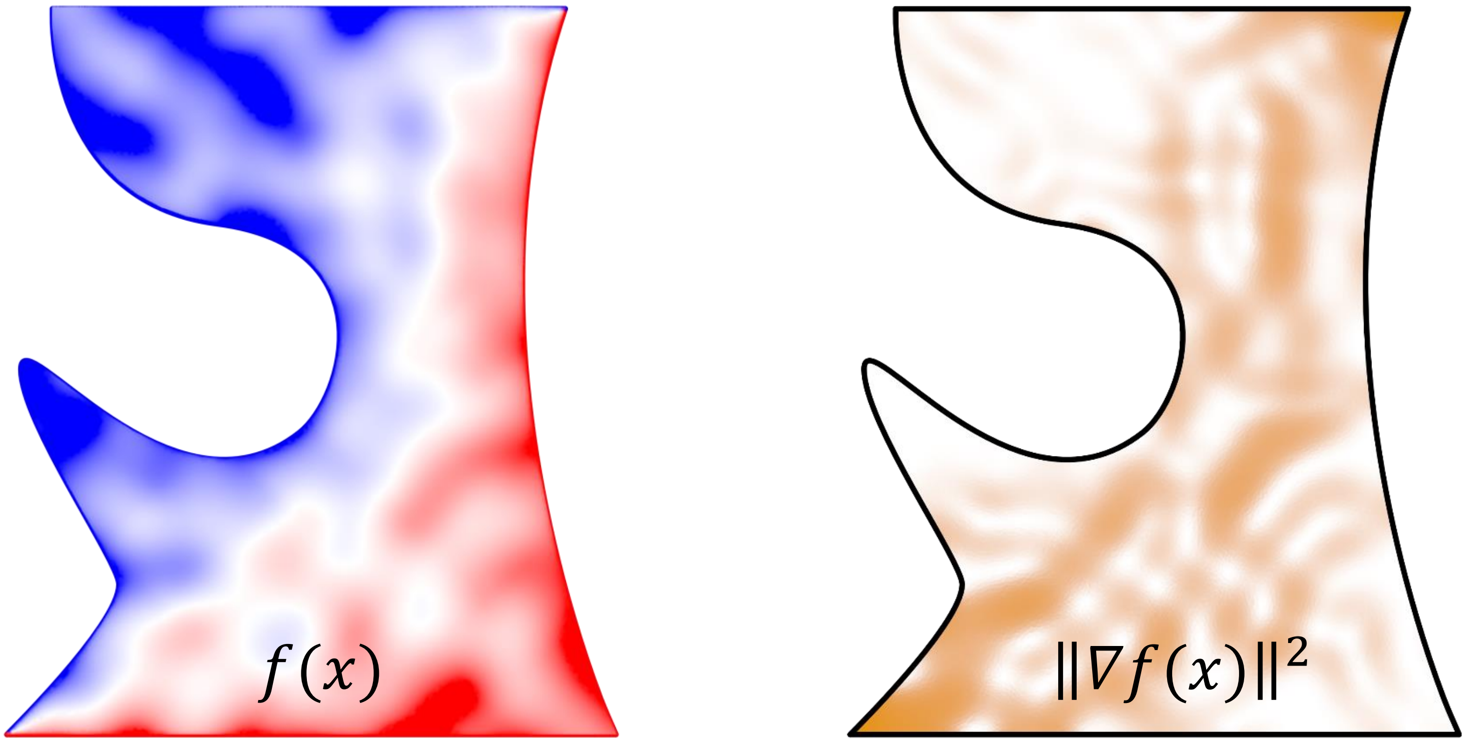
Don't want large variation over small distance



Desired: Smooth Functions



Dirichlet Energy



$$E(f) \equiv \int_\Omega \|\nabla f\|^2 \, dA$$

Dirichlet Energy

$$E(f) \equiv \int_\Omega \|\nabla f\|^2 \, dA$$

- Nonnegative
- Zero for constant functions
- Measures smoothness

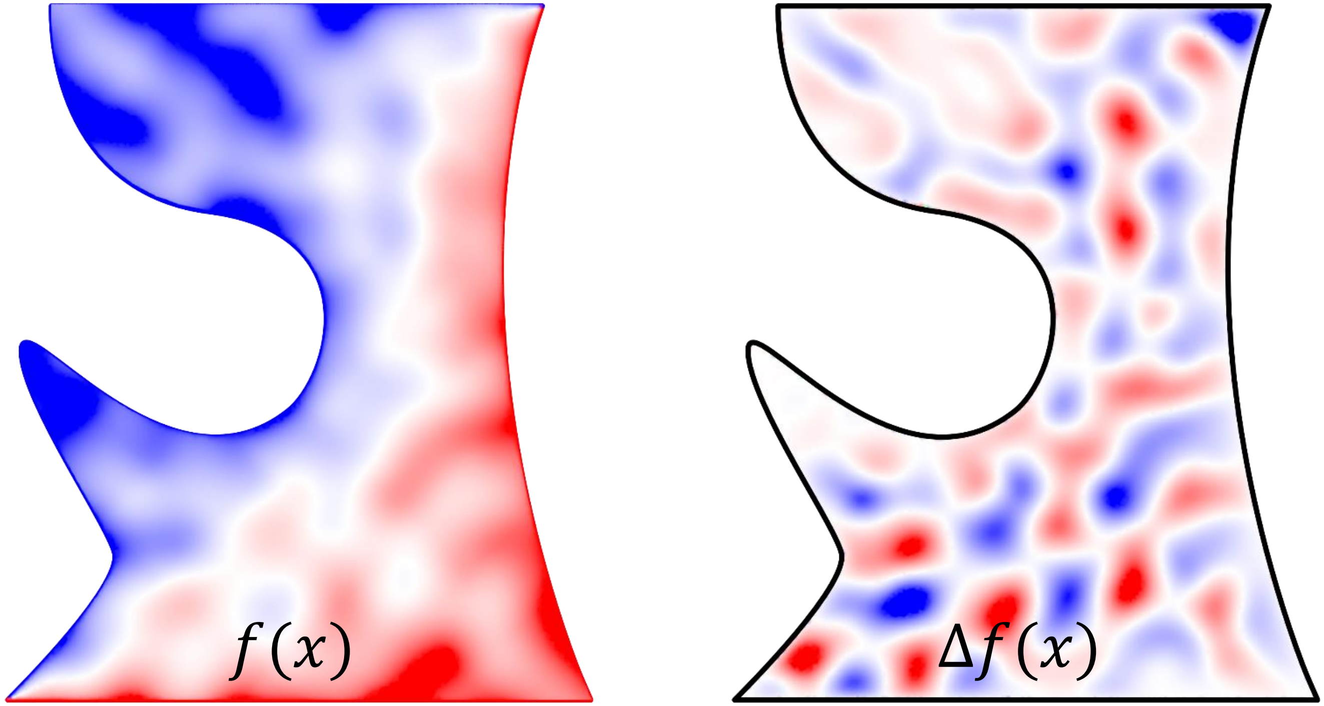
Dirichlet Problem

$$\begin{aligned} \min_f \quad & E(f) \equiv \int_\Omega \|\nabla f\|^2 \, dA \\ \text{s.t.} \quad & f|_{\partial\Omega} \text{ given} \end{aligned}$$

Set derivative to zero

$$\begin{aligned} \Delta f(x) &= 0 \quad \forall x \in \Omega \backslash \partial\Omega \\ f(x) &= f_0(x) \quad \forall x \in \partial\Omega \end{aligned}$$

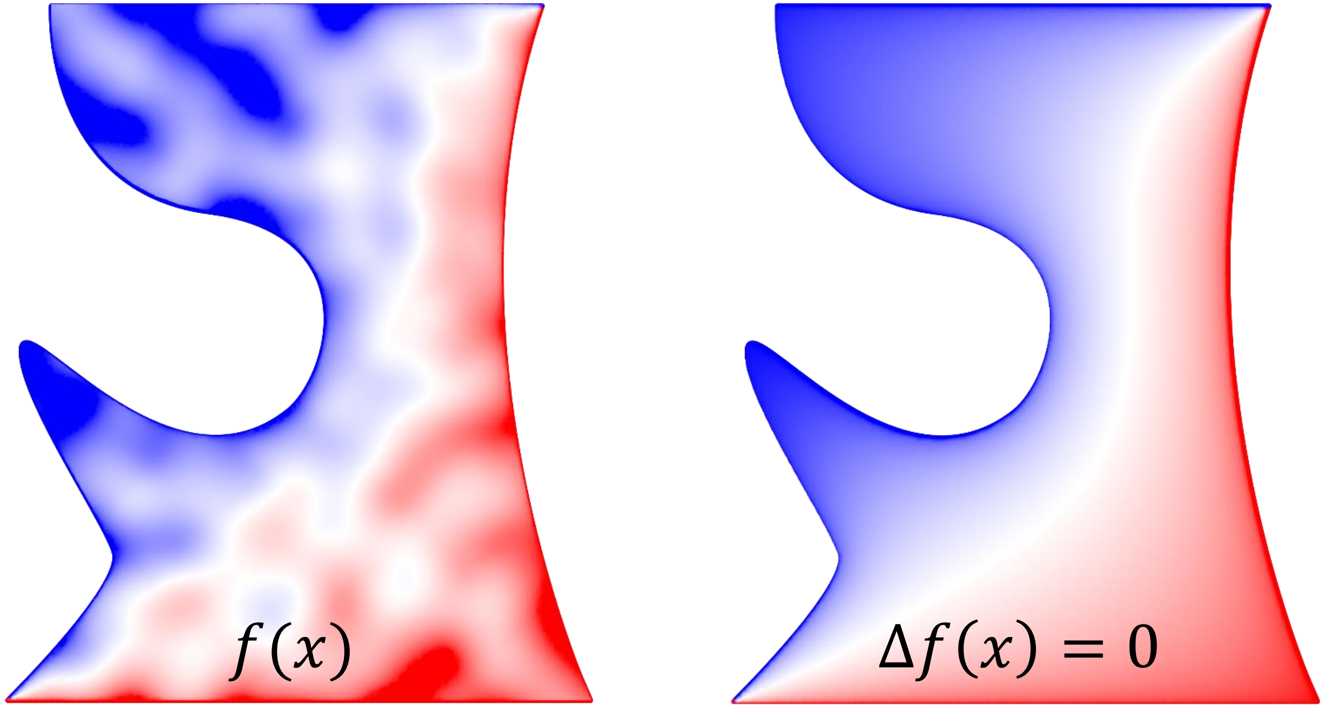
Laplacian Operator



$$\Delta f(x) = 0 \quad \forall x \in \Omega \backslash \partial\Omega$$

$$f(x) = f_0(x) \quad \forall x \in \partial\Omega$$

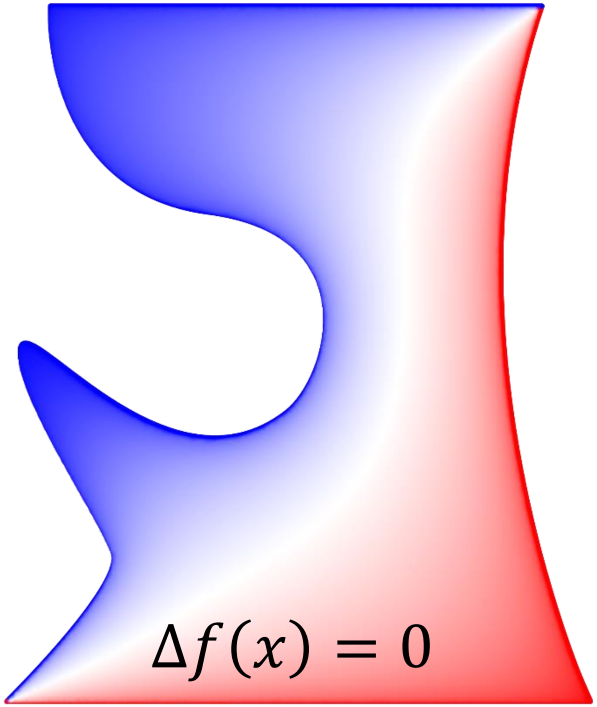
Laplacian Operator



$$\Delta f(x) = 0 \quad \forall x \in \Omega \backslash \partial\Omega$$
$$f(x) = f_0(x) \quad \forall x \in \partial\Omega$$

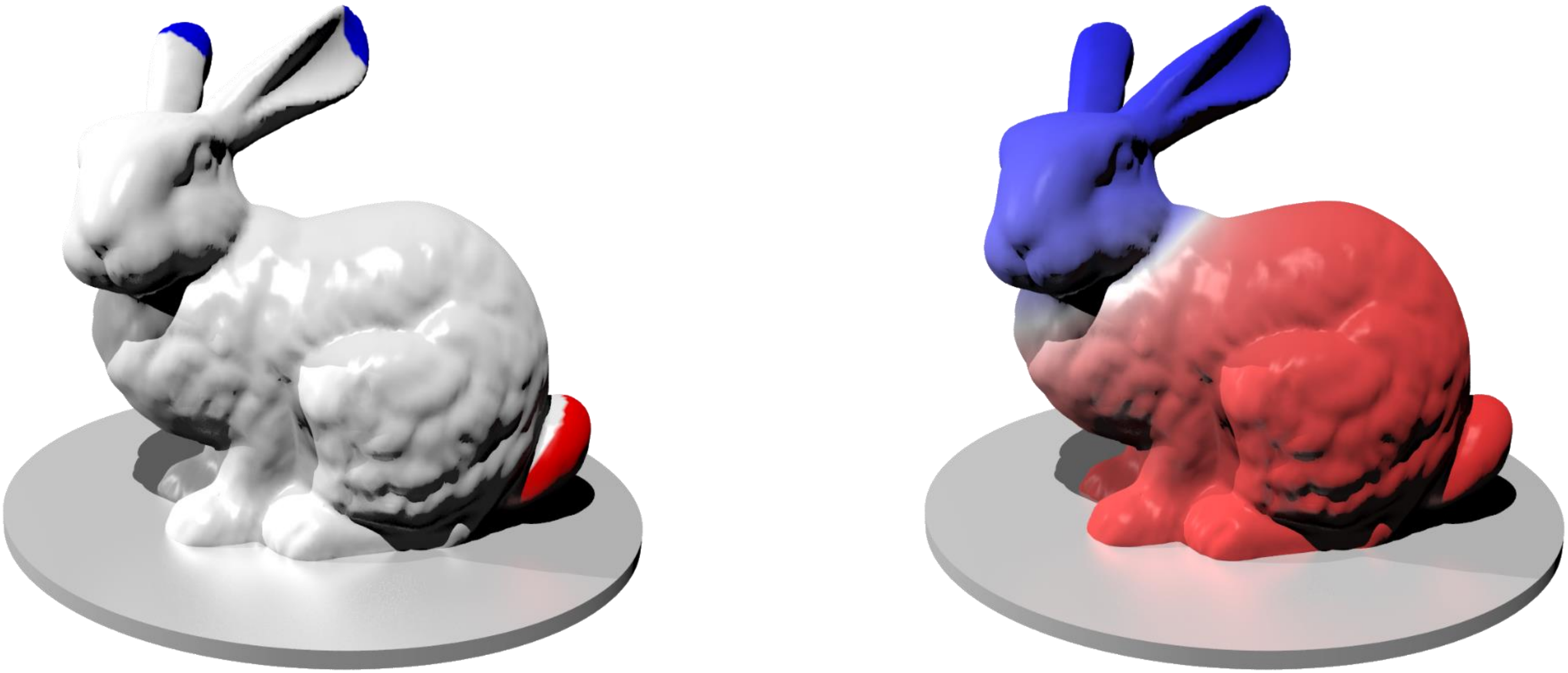
Intuition

Temperature at steady state



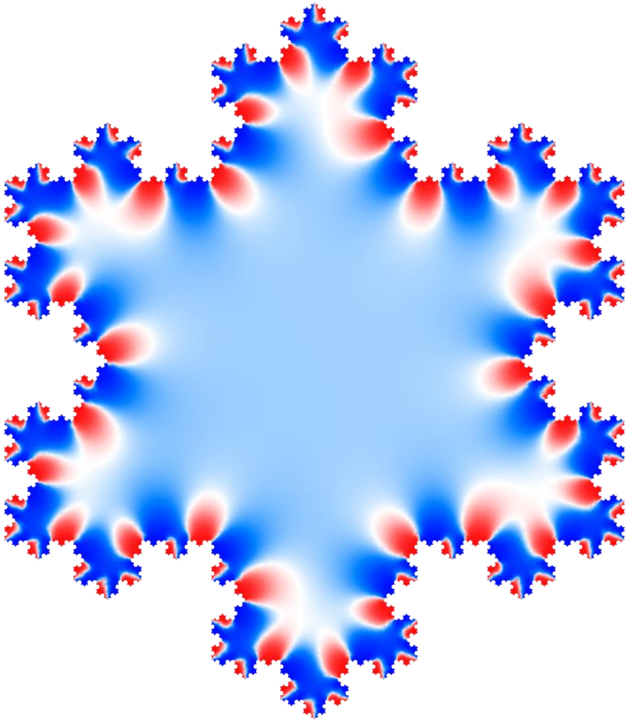
$$\Delta f(x) = 0 \quad \forall x \in \Omega \backslash \partial\Omega$$
$$f(x) = f_0(x) \quad \forall x \in \partial\Omega$$

Dirichlet on Other Domains



$$E(f) \equiv \int_{\Omega} \|\nabla f\|^2 \, dA$$

Dirichlet on Other Domains



$$E(f) \equiv \int_\Omega \|\nabla f\|^2 \, dA$$

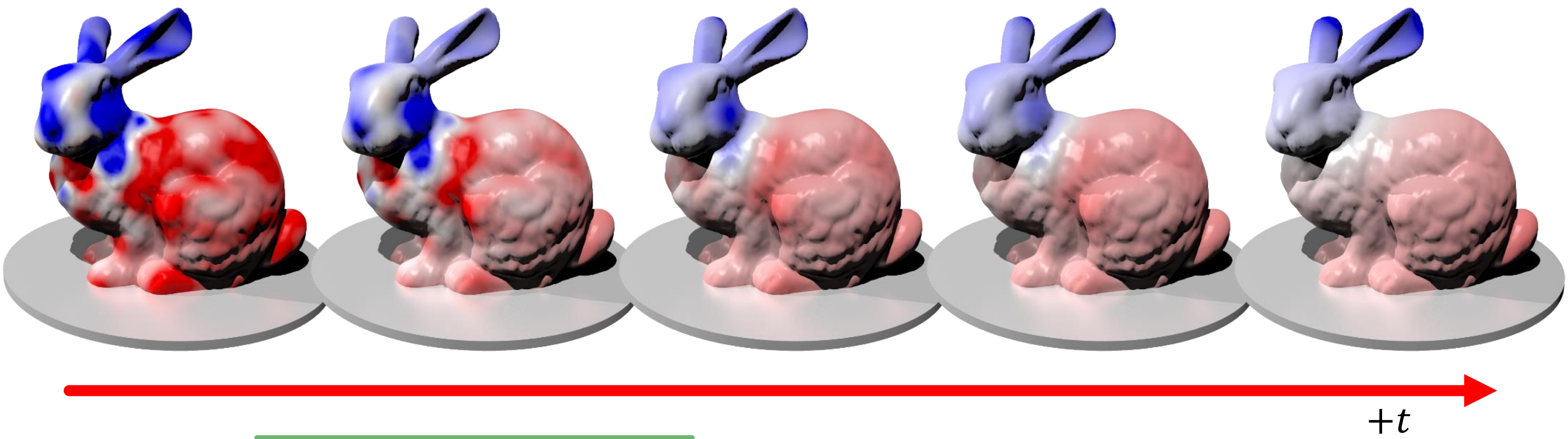
Pattern for Finding Laplacian

$$\begin{aligned} \min_f \quad & E(f) \equiv \int_\Omega \|\nabla f\|^2 \, dA \\ \text{s.t.} \quad & f|_{\partial\Omega} \text{ given} \end{aligned}$$

Set derivative to zero

$$\begin{aligned} \Delta f(x) = 0 \quad & \forall x \in \Omega \backslash \partial\Omega \\ f(x) = f_0(x) \quad & \forall x \in \partial\Omega \end{aligned}$$

Related Equations

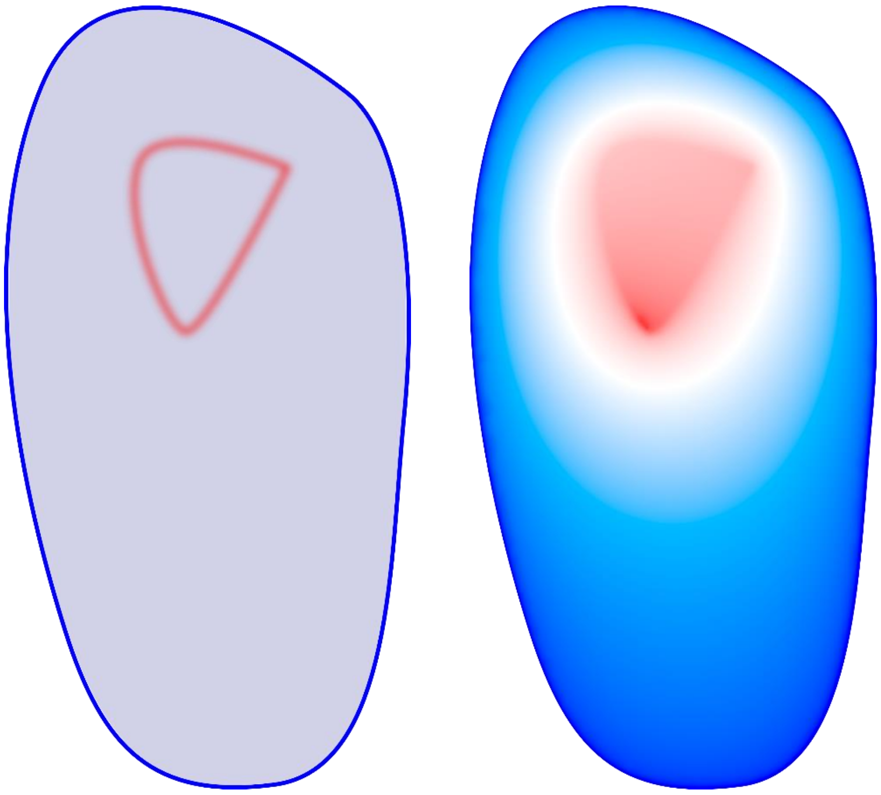


$$\frac{\partial f}{\partial t} = \Delta f$$

Gradient descent on Dirichlet energy

Heat equation

Related Equations



$$\Delta f = g$$

$$\min_f \int_\Omega \|\nabla f - v\|^2 \, dA$$
$$\implies \Delta f = \nabla \cdot v$$

Poisson equation

Algebraic Properties

- *linearity*: $\Delta\left(f(x)+\alpha g(x)\right)=\Delta f(x)+\alpha\Delta g(x)$
- *constants in kernel*: $\Delta\alpha=0$

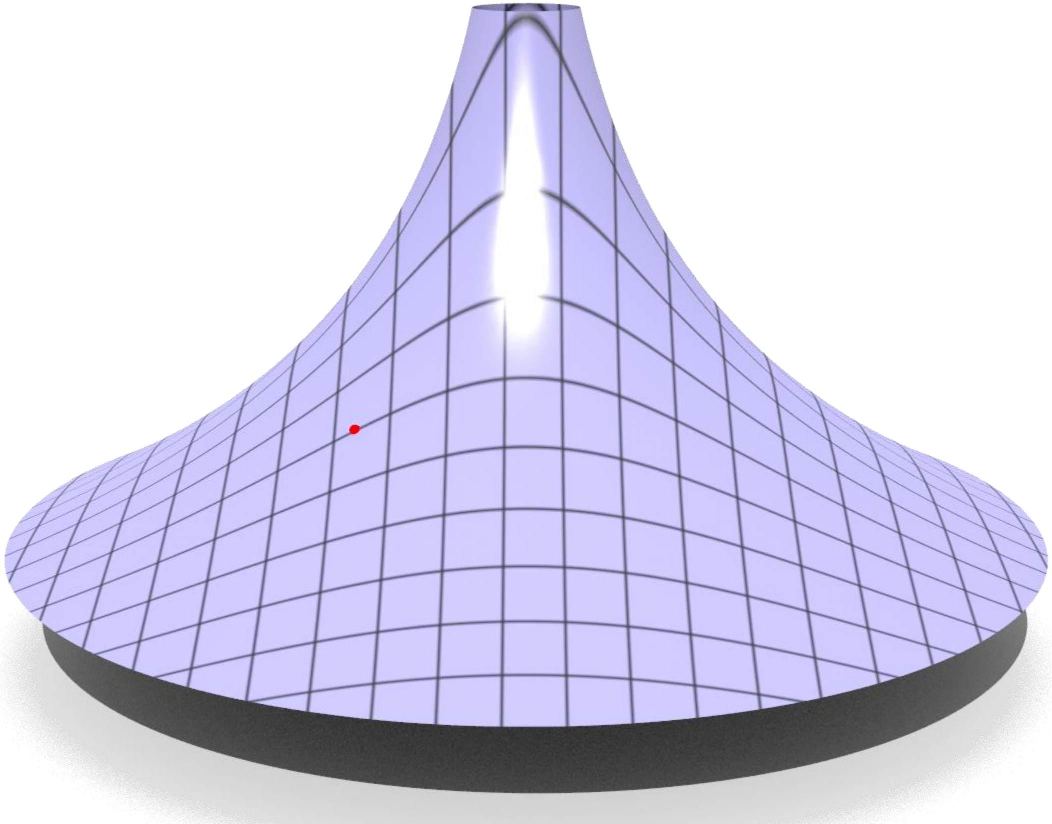
for functions that vanish on ∂M:

- *self-adjoint*: $\int_M f\Delta g\,dA=-\int_M\langle\nabla f,\nabla g\rangle\,dA=\int_M g\Delta f\,dA$
- *negative*: $\int_M f\Delta f\,dA\leq 0$

(intuition: $\Delta\approx$ an ∞-*dimensional* negative-semidefinite matrix)

Harmonic Functions

$$\Delta f(x) = 0$$



Harmonic Functions

$$\Delta f(x) = 0$$

- Smooth and analytic
- Mean value property:

$$f(x) = \frac{1}{\pi r^2} \int_{B_r(x)} f(y)\, dA$$

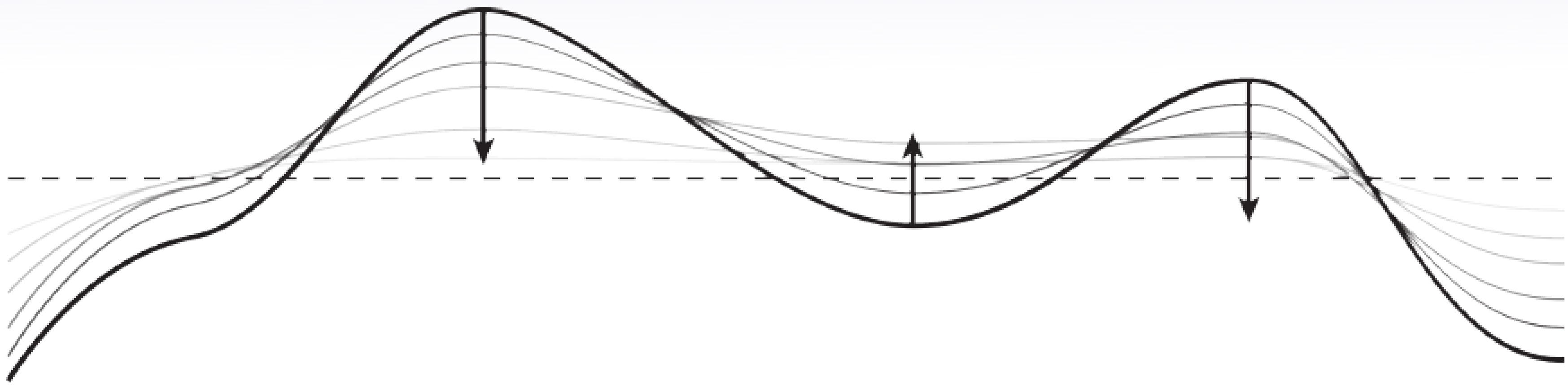
- Maximum principle: No local maxima or minima

(can have saddles)

Geometric Properties

for a curve $\gamma(u) = (x[u], y[u]) : \mathbb{R} \to \mathbb{R}^2$

- total Dirichlet energy $\int \|\nabla x\|^2 + \|\nabla y\|^2$ is arc length
- $\Delta\gamma = (\Delta x, \Delta y)$ is gradient of arc length
- $\Delta\gamma$ is the *curvature normal* $\kappa\hat{n}$
- minimal curves are harmonic (straight lines)

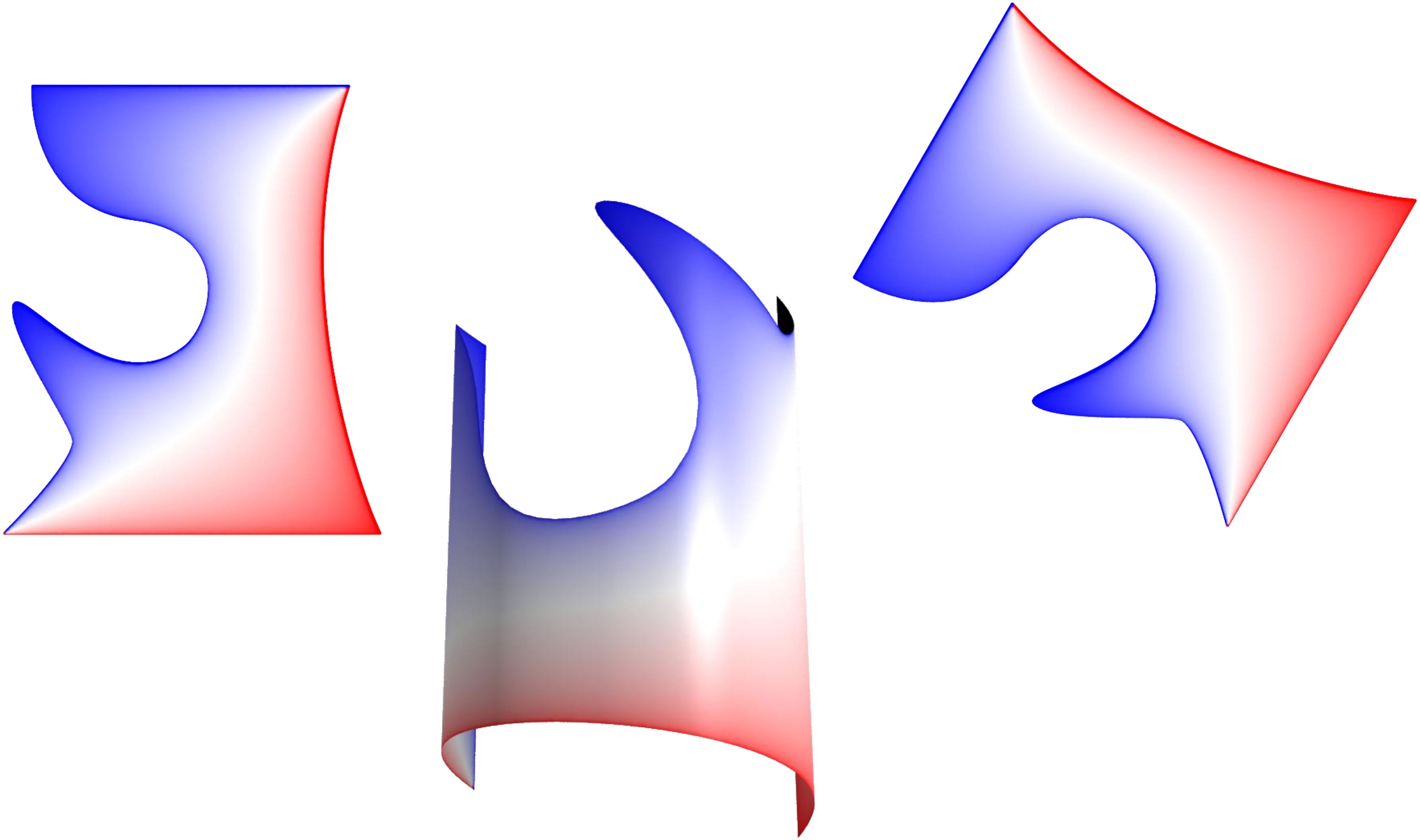


Geometric Properties

for a surface $r(u,v) = (x[u,v], y[u,v], z[u,v]) : \mathbb{R} \to \mathbb{R}^3$

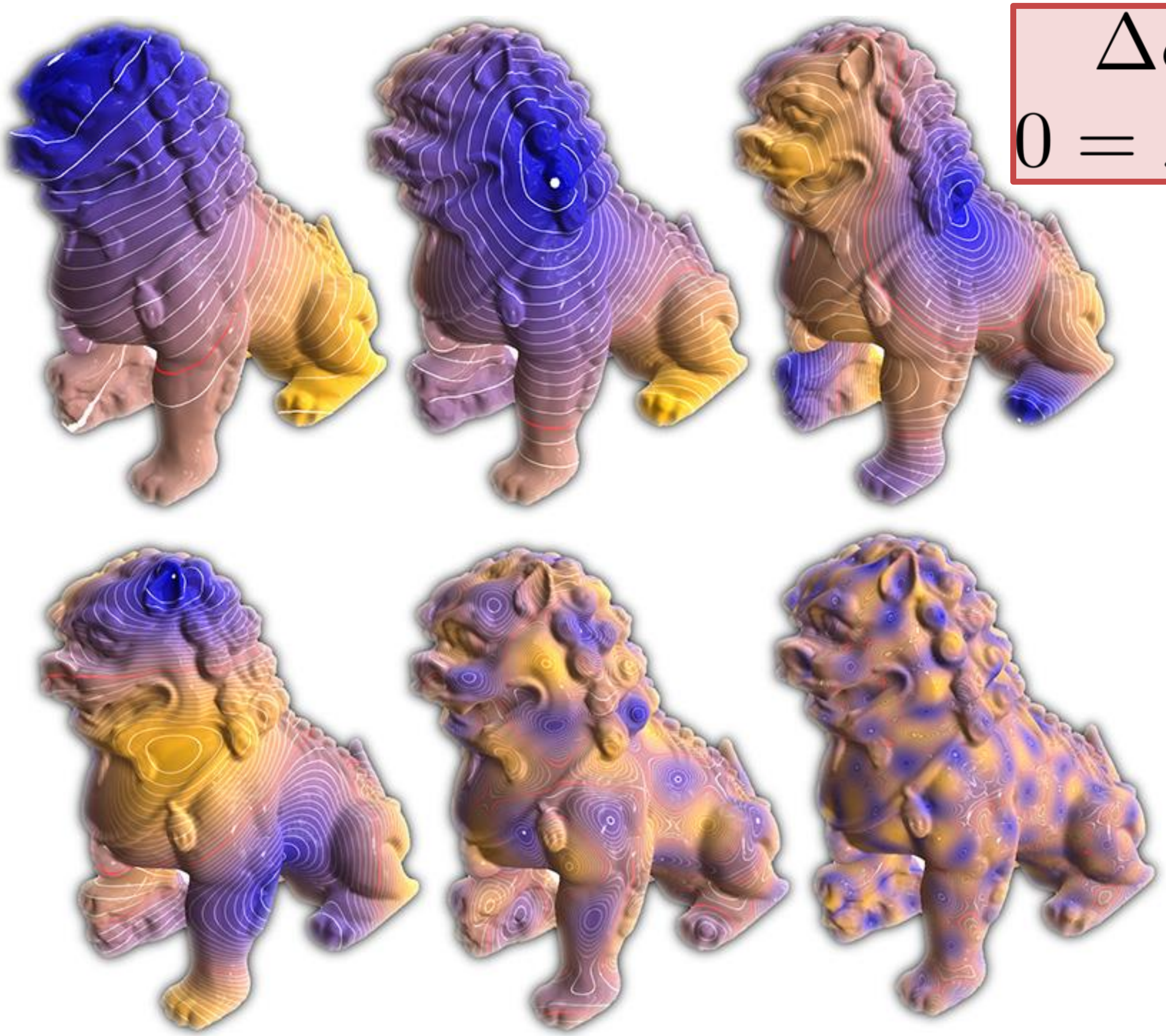
- total Dirichlet energy is surface area
- $\Delta r = (\Delta x, \Delta y, \Delta z)$ is gradient of surface area
- Δr is the *mean curvature normal* $2H\hat{n}$
- minimal surfaces are harmonic!

Laplacian is Intrinsic



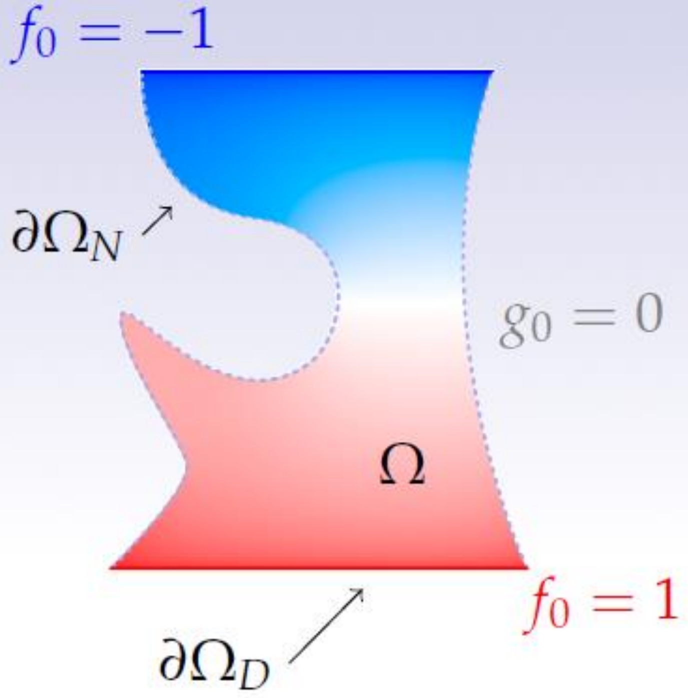
Laplacian Spectrum

$$\Delta\phi_k(x) = \lambda_k\phi_k(x)$$
$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$$



Intrinsic Fourier basis

Boundary Conditions



- can specify $\nabla f \cdot \hat{n}$ on boundary instead of f:

$$\begin{aligned} \Delta f(x) &= 0 && x \in \Omega \\ f(x) &= f_0(x) && x \in \partial\Omega_D && \textit{(Dirichlet bdry)} \\ \nabla f \cdot \hat{n} &= g_0(x) && x \in \partial\Omega_N && \textit{(Neumann bdry)} \end{aligned}$$

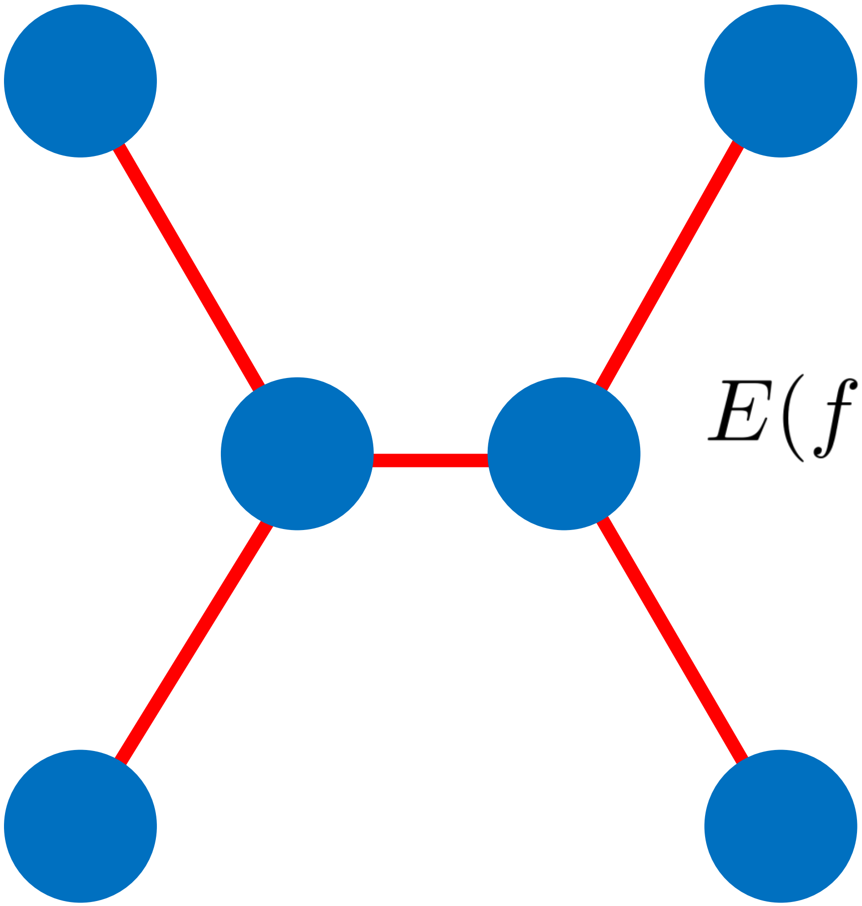
- usually: $g_0 = 0$ (*natural* bdry conds)
- physical interpretation: free boundary through which heat cannot flow

Numerical Linear Algebra

- Laplacian matrices should be:
  - Sparse
  - Positive (semi-)definite

- Typical solvers
  - Direct: LDLT
  - Iterative: Conjugate gradients

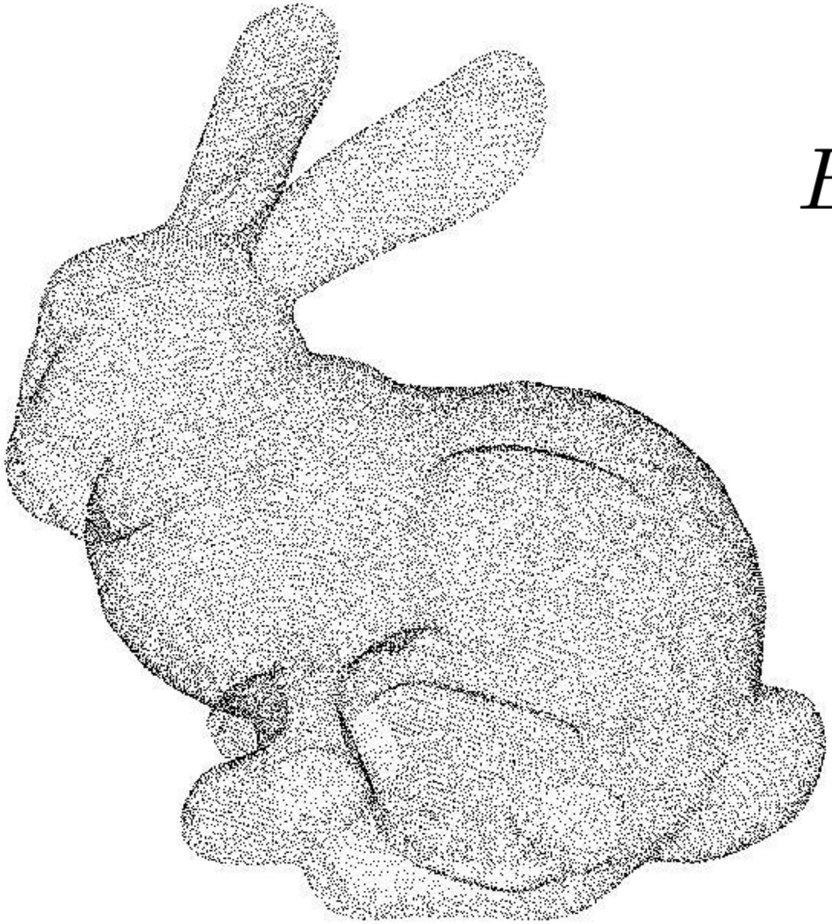
Constructing Laplacian Operators



$$E(f) = \sum_{e=(i,j)} (f(v_i) - f(v_j))^2$$

Per-vertex functions on a graph

Constructing Laplacian Operators

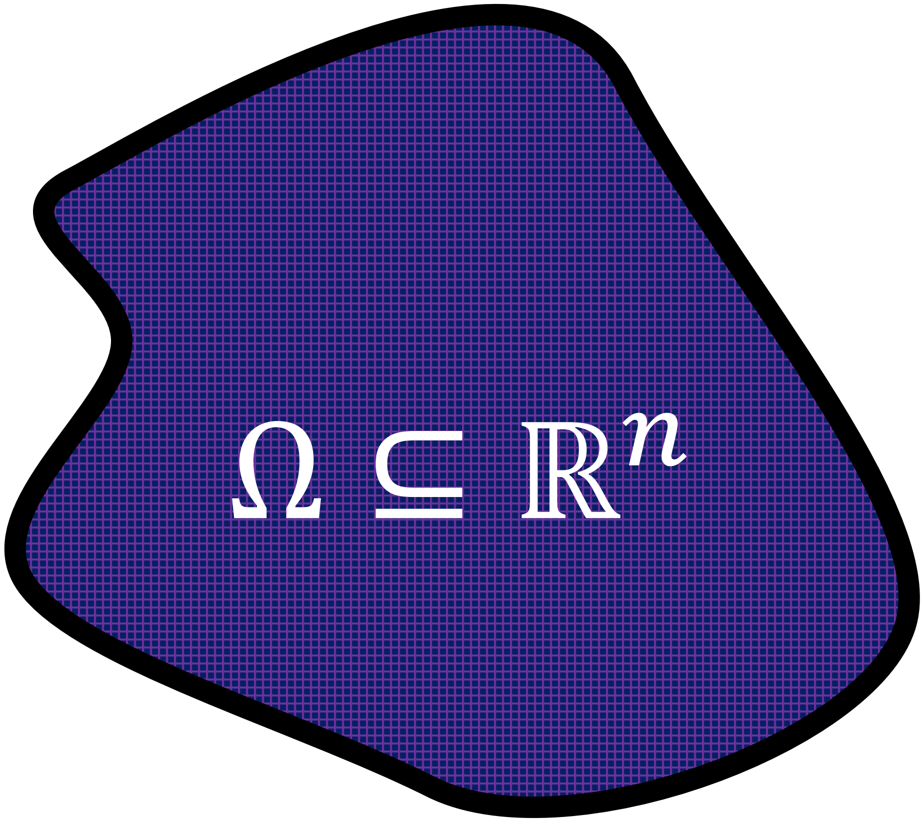


$$E(x) = \sum_{i,j} w_{ij}(x_i - x_j)^2$$

Given pairwise similarity measure

Integration by Parts

$$\int_\Omega f \Delta g \, dA = \text{boundary terms} - \int_\Omega \nabla f \cdot \nabla g \, dA$$



L^2 Dual of a Function

$$f : \Omega \to \mathbb{R}$$

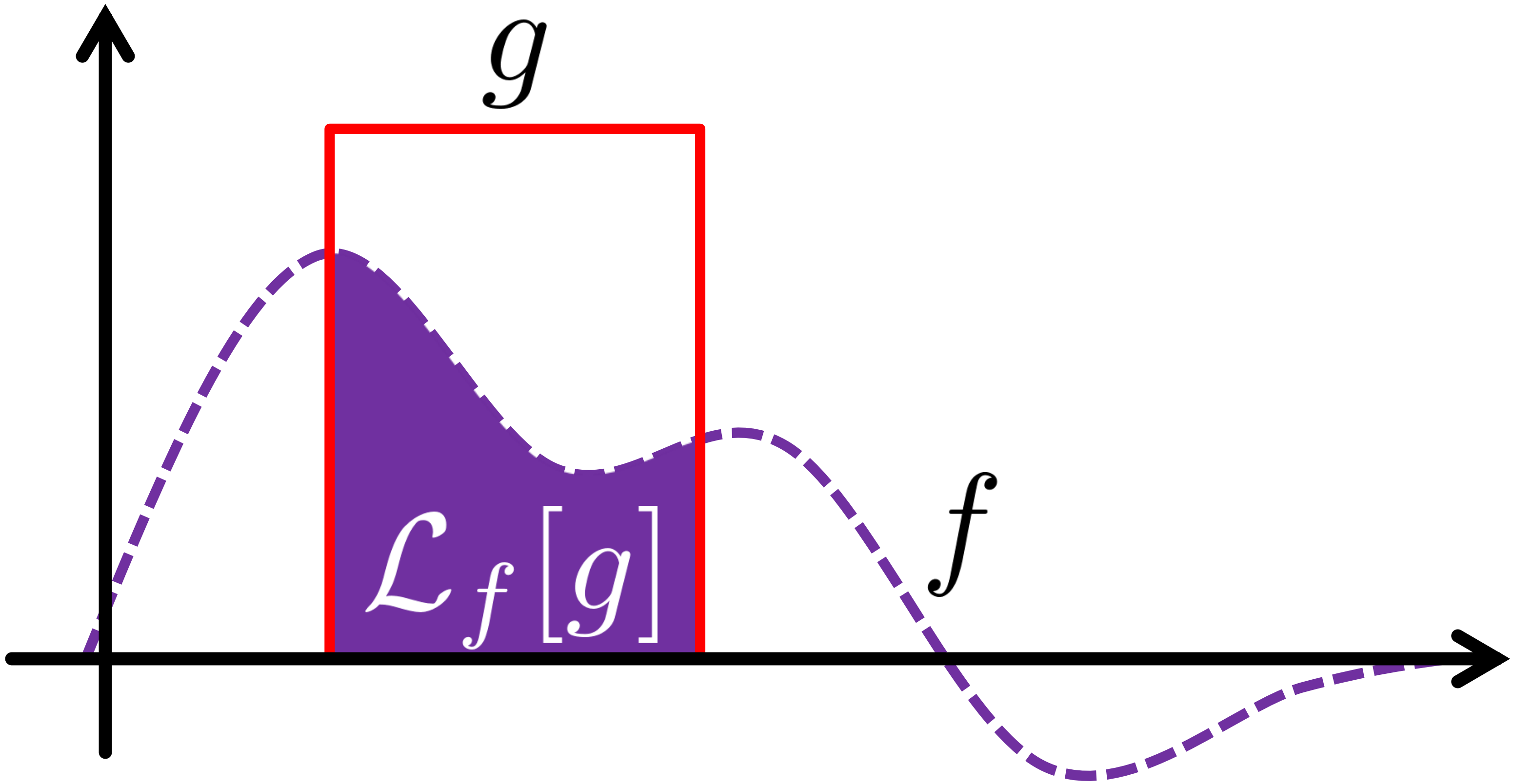
$$\Downarrow$$

$$\mathcal{L}_f : L^2(\Omega) \to \mathbb{R}$$

$$\mathcal{L}_f[g] \equiv \int_\Omega f g \, dA$$

↑ "Test function"

Observation



Can recover function from dual

Dual of Laplacian

$$\mathcal{L}_{\Delta f}[g] = \int_\Omega g \Delta f \, dA$$

$$= \text{const.} - \int_\Omega \nabla f \cdot \nabla g \, dA$$

One derivative is enough

Use Laplacian without evaluating it!

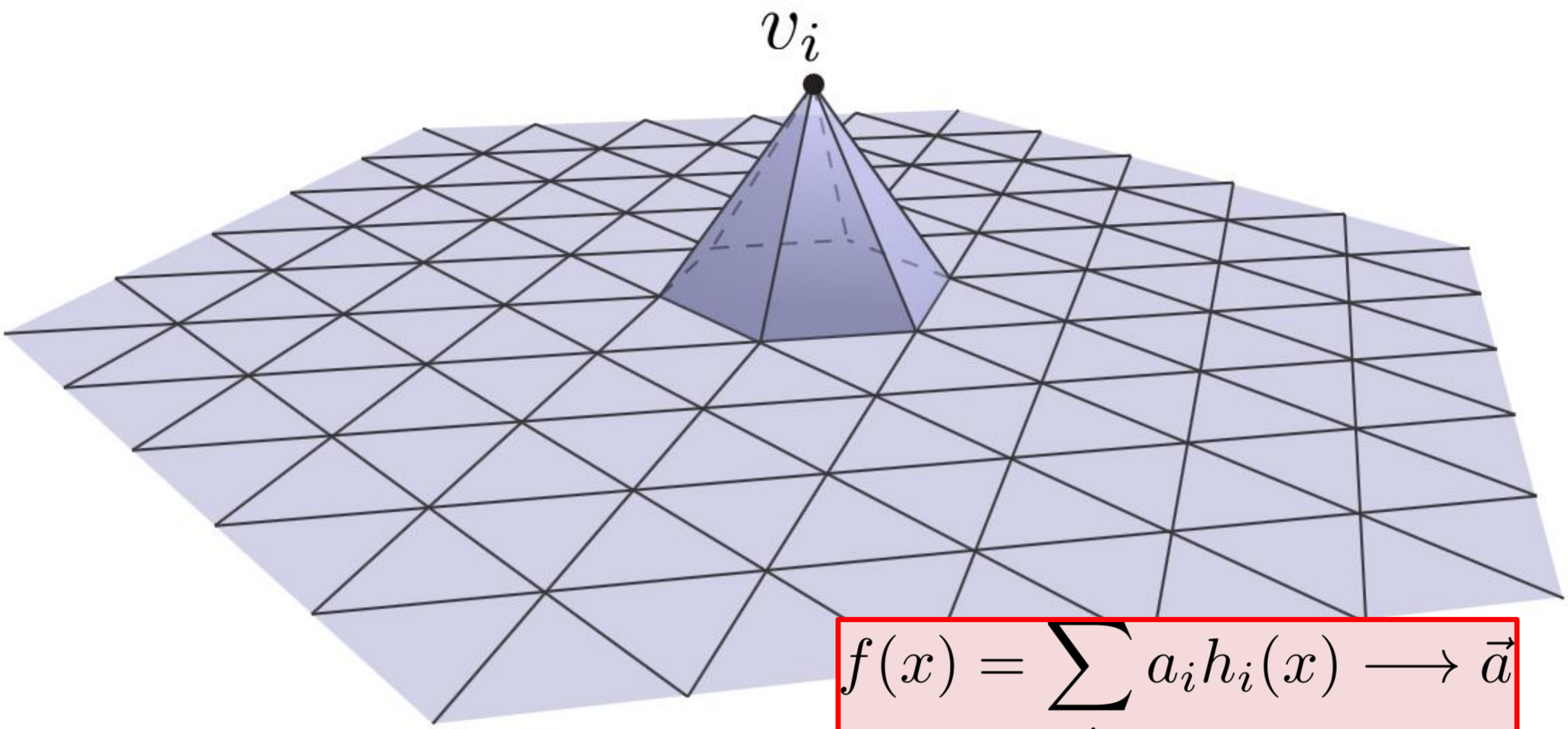
Galerkin's Approach

Choose one of each:

- Function space
- Test functions

Often the same!

First Order Finite Elements



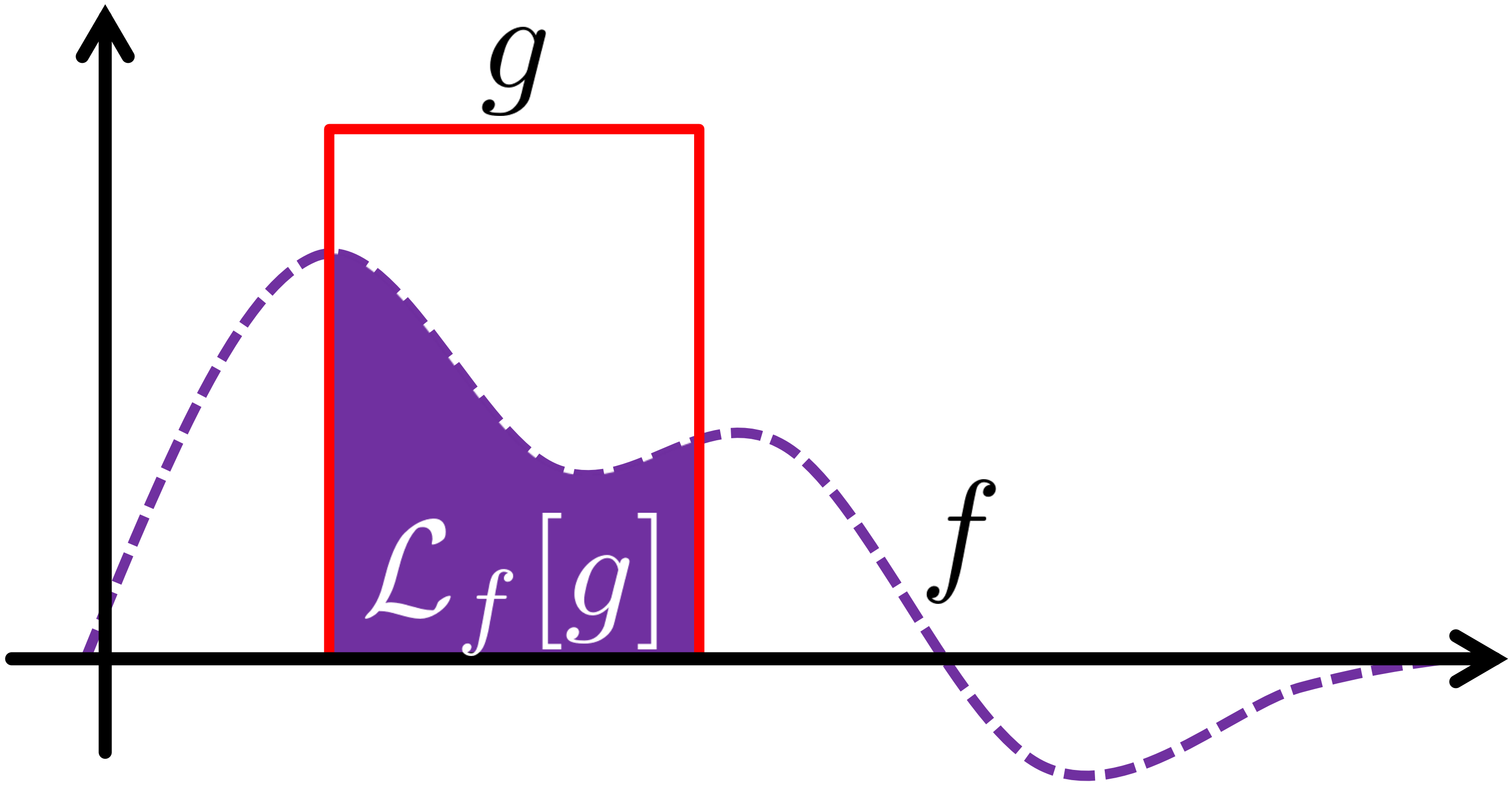
$$f(x) = \sum_i a_i h_i(x) \longrightarrow \vec{a}$$

Image by K. Crane

One "hat function" per vertex

Weak Solutions

$$\int_\Omega \phi \Delta f \, dA = \int_\Omega \phi g \, dA \; \forall \text{test functions } \phi$$



Finite Elements Weak Solutions

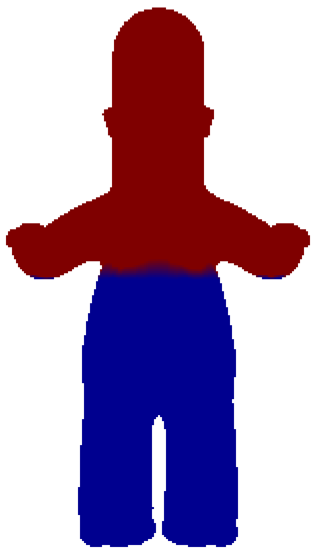
$$\int_\Omega h_i \Delta f \, dA = \int_\Omega h_i g \, dA \, \forall \text{hat functions } h_i$$

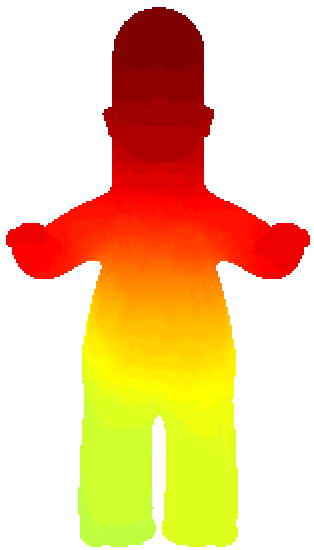
$$\begin{aligned}
\int h_i \Delta f \, dA &= -\int \nabla h_i \cdot \nabla f \, dA \\
&= -\int \nabla h_i \cdot \left(\sum_j a_j h_j \right) \, dA \\
&= -\sum_j a_j \int \nabla h_i \cdot \nabla h_j \, dA \\
&\equiv \sum_j L_{ij} a_j \text{ (linear system!)}
\end{aligned}$$

Poisson Equation with FEM

$\Delta f = g$ → $Lf = Ag$

g



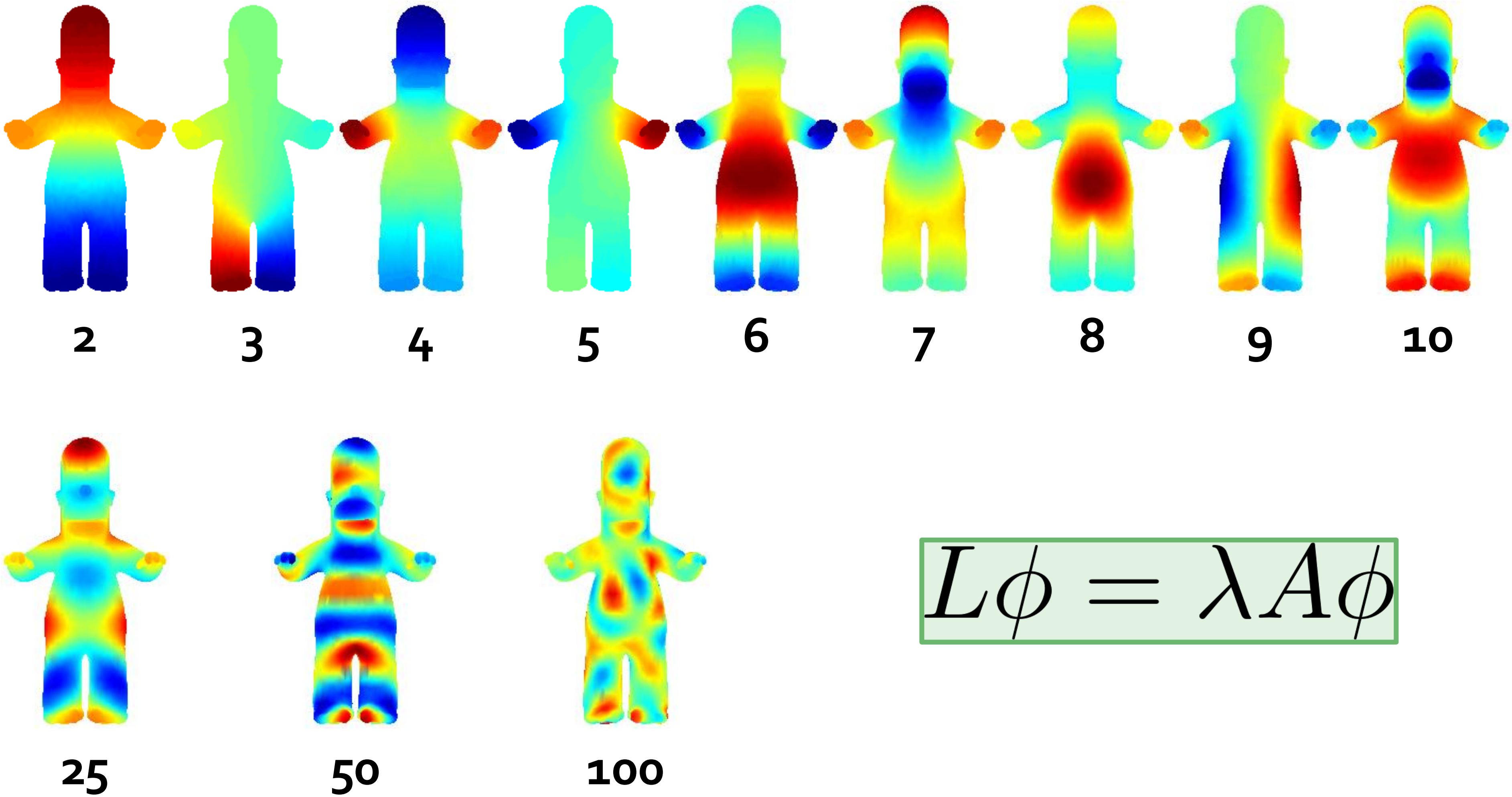


f

Must integrate to zero

Determined up to constant

(Generalized) Eigenhomers



$$L\phi = \lambda A\phi$$

Applications to Geometry

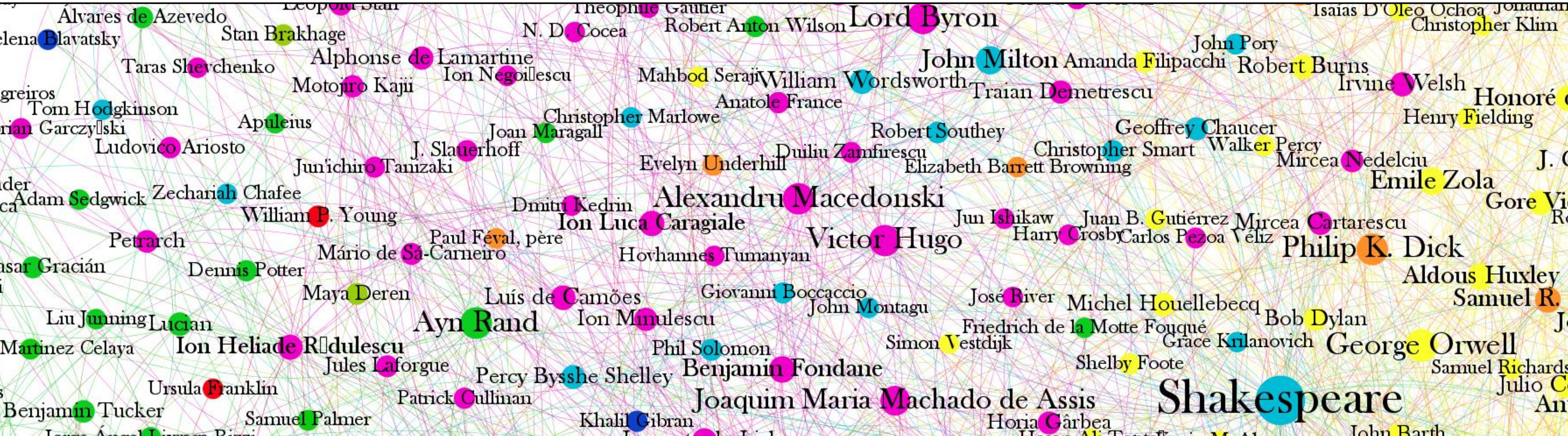
<other_slides>

Connections to Machine Learning

Most obvious:

Graph Laplacian

- Geometric structure for data points
- Use case: semi-supervised learning



Connections to Machine Learning

But:

Graph Laplacian is a weak notion of geometry.

- How do you construct the graph?
- How do you understand distances?

Connections to Machine Learning

Laplacians (and their inverses) come from "kernel matrices."

- Ingredients: Gradients and inner products
- PDE in high-dimensional point clouds?
- Can we learn Laplacians?
- Can we determine intrinsic dimensionality?
- Can you hear the shape of a dataset?

Image by K. Crane

Laplacian Operators

Headed out tomorrow!