Euler-Maclaurin Formula

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1 Introduction

Euler-Maclaurin summation formula is an important tool of numerical analysis. Simply put, it gives us an estimation of the sum $\sum_{i=0}^{n} f(i)$ through the integral $\int_0^n f(t)dt$ with an error term given by an integral which involves Bernoulli numbers. In the most general form, it can be written as

$$\begin{aligned}\sum_{n=a}^{b} f(n) &= \int_a^b f(t)dt + \frac{1}{2}(f(b)+f(a)) + \\ &+ \sum_{i=2}^{k} \frac{b_i}{i!}(f^{(i-1)}(b) - f^{(i-1)}(a)) - \\ &- \int_a^b \frac{B_k(\{1-t\})}{k!} f^{(k)}(t)dt\end{aligned} \tag{1}$$

where a and b are arbitrary real numbers with difference $b-a$ being a positive integer number, B_n and b_n are Bernoulli polynomials and numbers, respectively, and k is any positive integer. The condition we impose on the real function f is that it should have continuous k-th derivative. The symbol $\{x\}$ for a real number x denotes the fractional part of x. Proof of this theorem using $h-$calculus is given in the book [Ka] by Victor Kač. In this paper we would like to discuss several applications of this formula.

This formula was discovered independently and almost simultaneously by Euler and Maclaurin in the first half of the $XVIII$-th century. However, neither of them obtained the remainder term

$$R_k = \int_a^b \frac{B_k(\{1-t\})}{k!} f^{(k)}(t)dt \tag{2}$$

which is the most essential

Both used iterative method of obtaining Bernoulli's numbers b_i, but Maclaurin's approach was mainly based on geometric structure while Euler used purely analytic ideas. The remainder term was introduced later by S.D. Poisson. Further historical notes about this formula can be found in [Mi].

If $f(x)$ and all its derivatives tend to 0 as $x \to \infty$, the formula (1) can be simplified:

$$\sum_{n=a}^{\infty} f(n) = \int_a^{\infty} f(t)dt + \frac{1}{2}f(a) - \sum_{i=2}^{k} \frac{b_i}{i!} f^{(i-1)}(a) - \int_a^{\infty} \frac{B_k(\{1-t\})}{k!} f^{(k)}(t)dt \tag{3}$$

by letting $b \to \infty$ in the identity.

2 Preliminaries

The Bernoulli numbers b_n occur in a number of theorems of number theory and analysis. They can be defined by the following power series:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{b_n x^n}{n!}$$

or $b_n = \frac{d^n}{dx^n}(\frac{x}{e^x-1})\mid_{x=0}$. The first several Bernoulli numbers are the following (listed starting from b_0):

$$b_0 = 1, b_1 = -\frac{1}{2}, b_2 = \frac{1}{6}, b_3 = 0, b_4 = -\frac{1}{30}, b_5 = 0, b_6 = \frac{1}{42}, b_7 = 0,$$
$$b_8 = -\frac{1}{30}, b_9 = 0, b_{10} = \frac{5}{66}, b_{11} = 0, b_{12} = -\frac{691}{2730}, b_{13} = 0, b_{14} = \frac{7}{6}$$

The odd terms in the sequence are all 0 except the first one $b_1 = -\frac{1}{2}$ and they are all rational. This fact can be proved by considering the Taylor expansion of e^x. This implies that we can drop the odd terms in the summations in the formulas (1) and (3). As you see, there is no simple pattern in the distribution of the first few of these numbers. However, we know an asymptotic expansion of the Bernoulli numbers b_n when n is very large

$$b_{2n} \sim (-1)^{n-1} 4\sqrt{\pi n}(\frac{n}{\pi e})^{2n} \tag{4}$$

Bernoulli polynomials $B_n(x)$ are defined in a similar way for nonnegative integers n:

$$\frac{ze^{zx}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!} \tag{5}$$

Differentiating with respect to x, we get a relation:

$$B_n'(x) = nB_{n-1}(x)$$

Putting $x = 0$ into (5), gives us $B_n(0) = b_n$.

It is useful to define for a positive integer n, *the periodic Bernoullian function* $\bar{B}_n(x) = B_n(\{x\})$, where $\{x\}$ denotes the fractional part of x. Clearly, $\bar{B}_n$ is periodic with period 1 and continuous on $[0, 1)$. It is not hard to show that $\bar{B}_n$ satisfies the two properties above, i.e.

(i) $\bar{B}_n(0) = b_n$; in addition, $\bar{B}_0(x) = 1$ for any x;

(ii) $\bar{B}_n'(x) = n\bar{B}_{n-1}(x)$, or equivalently, $\bar{B}_n(x) = n\int_0^x \bar{B}_{n-1}(t)dt + b_n$ for a positive integer n and $x \in [0, 1)$.

These two properties actually uniquely determine the function $\bar{B}_n(x)$ by inductive definition.

Remark that by substituting $x = 1$ into (5), one can obtain $B_n(0) = B_n(1)$ for $n > 1$. Therefore, the function $\bar{B}_n(x)$ is continuous on the whole real line for $n > 1$.

Consider the polynomial $P_k(x) = \frac{1}{k!}\bar{B}_k(x)$. This polynomial appears in the error term R_k in the formula (2), i.e. we can write it as

$$R_k = \int_a^b P_k(1-t)f^{(k)}(t)dt \tag{6}$$

Lemma 1. *For any nonnegative integer k and $x \in [0,1]$,*

$$P_{2k}(x) = (-1)^{k-1} \sum_{n=1}^{\infty} \frac{2\cos(2n\pi x)}{(2n\pi)^{2k}} \tag{7}$$

$$P_{2k+1}(x) = (-1)^{k-1} \sum_{n=1}^{\infty} \frac{2\sin(2n\pi x)}{(2n\pi)^{2k+1}} \tag{8}$$

Since $P_0(x)$ may not converge by (7), we assume $P_0(x) = 1$.

Proof. Recall that the series $\sum_{n=1}^{\infty} \frac{1}{n^m}$ converges for any $m > 1$. Then the series on the right hand sides of the equations above converge by the comparison test, since absolute values of sine and cosine functions are bounded by 1. Consider the function $P_n^*(x)$ defined by these series. It is periodic with period 1. Let us differentiate $P_n^*(x)$:

$$\frac{d}{dx} P_{2k+1}^*(x) = (-1)^{k-1} \sum_{n=1}^{\infty} \frac{4n\pi\cos(2n\pi x)}{(2n\pi)^{2k+1}} = P_{2k}^*(x)$$

$$\frac{d}{dx} P_{2k}^*(x) = (-1)^{k-2} \sum_{n=1}^{\infty} \frac{4n\pi\sin(2n\pi x)}{(2n\pi)^{2k}} = P_{2k-1}^*(x)$$

Thus, $n!P_n^*$ do satisfy the property (ii) of $\bar{B}_n$. Let us check the values of $P_n^*(x)$ at $x = 0$.

$$P_{2k}^*(0) = (-1)^{k-1} \sum_{n=1}^{\infty} \frac{2}{(2n\pi)^{2k}}$$

$$P_{2k+1}^*(0) = 0$$

Lemma 2 (Euler)**.** *For a positive integer k, the zeta function $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$ satisfies*

$$\zeta(2k) = \frac{(-1)^{k-1}(2\pi)^{2k} b_{2k}}{2(2k)!} \tag{9}$$

Euler proved this identity by comparing two different power series expansions of $x \cot x$. A more elementary proof is given in Papadimitriou's paper, and is described in a very accessible way in the article by T. Apostol [Ap].

This lemma gives us that $P_{2k}^*(0) = \frac{2(-1)^{k-1}}{(2\pi)^{2k}}\zeta(2k) = \frac{b_{2k}}{(2k)!}$. Therefore, for a positive integer n, we have

$$P_n^*(0) = \frac{b_n}{n!}$$

Since $P_1^*(x) = -\sum_{n=1}^{\infty} \frac{\sin(2n\pi x)}{n\pi} = x - \frac{1}{2}$, so $P_1^*(x) = P_1(x)$ for $x \in [0,1)$ by the Fourier expansion of the polynomial.

Therefore, both defining properties (i) and (ii) of $P_n(x) = \frac{1}{n!}\bar{B}_n(x)$ are satisfied. Thus, $P_n^*(x) = P_n(x)$.

$\square$

Corollary 1. *For a nonnegative integer k,*

$$P_{2k}(1-x) = P_{2k}(x)$$

$$P_{2k+1}(1-x) = -P_{2k+1}(x)$$

Proof. It follows right away from the series given in the lemma. □

Finally, we can rewrite our original Euler-Maclaurin formula as follows:

$$\begin{aligned}\sum_{n=a}^{b} f(n) &= \int_a^b f(t)dt + \frac{1}{2}(f(b)+f(a)) + \sum_{i=1}^{k} \frac{b_{2i}}{(2i)!}(f^{(2i-1)}(b) - f^{(2i-1)}(a)) + \\ &\quad + \int_a^b P_{2k+1}(t) f^{(2k+1)}(t)dt\end{aligned} \tag{10}$$

where k is a nonnegative integer.

The level of approximation in the Euler-Maclaurin formula largely depends on the asymptotic behavior of the remainder term R_k. It may happen that R_k goes to zero while k goes to infinity. In this case, we would get

$$\sum_{n=a}^{b} f(n) = \int_a^b f(t)dt + \frac{1}{2}(f(b)+f(a)) + \sum_{i=1}^{\infty} \frac{b_{2i}}{(2i)!}(f^{(2i-1)}(b) - f^{(2i-1)}(a))$$

This formula gives a nice expression of the integral of the function of f in terms of a series of its values. Unfortunately, this happens quite rarely. Indeed, by (4), Bernoulli numbers b_{2i} increase very rapidly, so rapidly that the sequence $\frac{b_{2i}}{(2i)!}$ still grows fast. But the sequence $\frac{b_{2i}}{(2i)!}(f^{(2i-1)}(b) - f^{(2i-1)}(a))$ should converge to 0 as terms of a converging series. Therefore, $(2i-1)$-th derivative of f has to attain very small values to counterbalance growth of the Bernoulli numbers. In fact, most functions occurring in practical applications do not have this property as justified in [Kn]. Consequently, valuable results of practical importance can be obtained by letting b go to infinity, but keeping k in (10) fixed.

3 Euler Constant

One of the most interesting applications is obtained when we consider the summation formula for the function $f(x) = \frac{1}{x}$. For a positive integer n, set $a = 1$, $b = n$, and $k = 1$. Substituting these values into our formula (10), we get

$$\sum_{i=1}^{n} \frac{1}{i} = \log n + \frac{1}{2n} + \frac{1}{2} + \int_1^n \frac{P_1(t)}{t^2} dt$$

Let us collect all error terms into $R(n)$ to obtain the following formula:

$$\sum_{i=1}^{n} \frac{1}{i} - \log n = R(n)$$

$$R(n) = \frac{1}{2n} + \frac{1}{2} + \int_1^n \frac{P_1(t)}{t^2} dt$$

Recall $P_1(x) = \bar{B}_1(x) = \{x\} - \frac{1}{2}$, so its absolute value is bounded by $\frac{1}{2}$. Hence, the error term $R(n)$ converges when $n \to \infty$, because $\int_1^\infty \frac{1}{t^2} = 1$. Let us denote by γ the limit $\lim_{n\to\infty} R(n)$.

An alternative way of proving existence of this constant involves usage of the following theorem, found in [Mw]:

Theorem 1 (Maclaurin-Cauchy)**.** *If $f(x)$ is positive, continuous, and tends monotonically to 0, then an Euler constant γ_f, which is defined below, exists*

$$\gamma_f = \lim_{n\to\infty} (\sum_{i=1}^n f(i) - \int_1^n f(x)dx)$$

Proof. Continuity of f guarantees existence of the integral $I_n = \int_1^n f(x)dx$ for any positive integer n. Since f is decreasing, we know maximum and minimum values of f over any closed interval:

$$\inf\{f(x) \mid x \in [k, k+1], k \in \mathbb{Z}_+\} = f(k+1) \tag{11}$$

$$\sup\{f(x) \mid x \in [k, k+1], k \in \mathbb{Z}_+\} = f(k) \tag{12}$$

Hence, we have the following inequalities:

$$f(k+1) \leq \int_k^{k+1} f(x)dx \leq f(k)$$

Summing these inequalities from $k = 1$ to $k = n-1$, we get

$$\sum_{i=2}^n f(i) \leq I_n \leq \sum_{i=1}^{n-1} f(i)$$

Hence, the difference $a_n = \sum_{i=1}^n f(i) - I_n$ satisfies $0 \leq f(n) \leq a_n \leq f(1)$, and

$$a_{n+1} - a_n = f(n+1) - \int_n^{n+1} f(x)dx \leq 0$$

So the sequence a_n is monotonically decreasing and is bounded below, so it converges to some number γ_f. □

Corollary 2. *The constant γ, representing the limit of difference between partial harmonic series sum and logarithm, is just $\gamma_{\frac{1}{x}}$.*

Due to its importance, this constant γ bears the name of *Euler-Mascheroni.* With high precision it is equal to

$$\gamma = 0.577215664901532860606512090 0\ldots$$

according to [Mw]. It was discovered by Euler, and it was computed by him up to its first 16 decimal digits. Later, Mascheroni improved his result by computing 32 decimal digits (although only first 19 were correct.) Euler-Mascheroni constant occurs in variety of areas of number theory and analysis; for example, it is conjectured that the number of Mersenne primes M_p not greater than x is about

$$\frac{e^\gamma}{\log 2}\log\log x$$

However, the question whether γ is rational or not is still an open problem despite the centuries passed! It is believed to be irrational and even transcendental by many mathematicians but there is no proof known up to date.

Let consider a more general type of functions, and instead of $f(x)=\frac{1}{x}$, we substitute $f(x)=\frac{1}{x^s}$ for $s>1$ into (3) with odd $k=2m+1$. We can do that since all derivatives of f as well as f itself tend to 0. We obtain:

$$\sum_{i=1}^{\infty}\frac{1}{i^s}=\frac{1}{s-1}+\frac{1}{2}-\sum_{i=1}^{m}\frac{b_{2i}}{(2i)!}f^{(2i-1)}(1)-\int_1^\infty P_{2m+1}(1-t)f^{(2m+1)}(t)dt$$

Note that $f^{(i)}(x)=(-1)^i\frac{s(s+1)\dots(s+i-1)}{x^{s+i}}=(-1)^i i!\binom{s+i-1}{i}\frac{1}{x^{s+i}}$. Therefore,

$$\sum_{i=1}^{\infty}\frac{1}{i^s}=\frac{1}{s-1}+\frac{1}{2}+\sum_{i=1}^{m}\binom{s+2i-2}{2i-1}\frac{b_{2i}}{2i}+(2m+1)!\binom{s+2m}{2m+1}\int_1^\infty\frac{P_{2m+1}(1-t)}{t^{s+2m+1}}dt$$

The expression on the left hand side is an important *Riemann zeta function*. Thus, we proved the following remarkable formula:

Theorem 2. *For any positive integer m and a real number $s>1$,*

$$\zeta(s)=\frac{1}{s-1}+\frac{1}{2}+\sum_{i=1}^{m}\binom{s+2i-2}{2i-1}\frac{b_{2i}}{2i}-(2m+1)!\binom{s+2m}{2m+1}\int_1^\infty\frac{P_{2m+1}(t)}{t^{s+2m+1}}dt$$

4 Stirling's formula

Another interesting application of the formula (10) is the Stirling's approximation formula. Let us take $f(x)=\log x$, $k=0$, $a=1$, and $b=n$, where n is any positive integer:

$$\sum_{i=1}^{n}\log i=(n\log n-n+1)+\frac{\log n}{2}+\int_1^n\frac{P_1(t)}{t}dt$$

Simplifying and extracting the error term $R(n)$, we get

$$\log n! = (n+\frac{1}{2})\log n-(n-1)+\int_1^n\frac{P_1(t)}{t}dt \quad (13)$$

$$R(n)=\log n!-(n+\frac{1}{2})\log n+n=\int_1^n\frac{P_1(t)}{t}dt+1 \quad (14)$$

Integration by parts gives us

$$\int_1^n\frac{P_1(t)}{t}dt=\frac{P_2(t)}{t}\Big|_1^n+\int_1^n\frac{P_2(t)}{t^2}dt$$

But $P_2(x) = \sum_{i=1}^{\infty} \frac{2\cos(2i\pi x)}{(2i\pi)^2}$, so using that the absolute value of cosine function is at most 1 and the series $\sum_{i=1}^{\infty} \frac{1}{i^2}$ converges, we see that absolute value of $P_2(x)$ is bounded. Then the integral above converges when $n \to \infty$:

$$C = \lim_{n\to\infty} R(n)$$

We can obtain the exact value of C in the following way. We have

$$\begin{aligned} 2\log(2 \cdot 4 \cdots 2n) &= 2n\log 2 + 2\log n! \\ &= 2n\log 2 + (2n+1)\log n - 2n + 2R(n) \\ &= (2n+1)\log 2n - 2n - \log 2 + 2R(n) \end{aligned} \tag{15}$$

On the other hand,

$$\log(2n+1)! = (2n+\frac{3}{2})\log(2n+1) - (2n+1) + R(2n+1) \tag{16}$$

Subtracting the second expression from the first,

$$\begin{aligned} \log\frac{2\cdot 4\cdots 2n}{1\cdot 3\cdots(2n+1)} &= (2n+1)\log\frac{2n}{2n+1} - \frac{1}{2}\log(2n+1) + 1 - \log 2 + 2R(n) - R(2n+1) \\ &= -\log(1+\frac{1}{2n})^{2n+1} - \frac{1}{2}\log(2n+1) + 1 - \log 2 + 2R(n) - R(2n+1) \end{aligned}$$

We are to use the *Wallis' product* formula,

Lemma 3 (Wallis)**.**

$$\lim_{n\to\infty} \frac{2\cdot 4\cdots 2n}{1\cdot 3\cdots(2n+1)}\frac{1}{\sqrt{n}} = \sqrt{\pi}$$

One way to derive this identity is to use the following well-known infinite product expansion of $\sin\pi x$ for $x = \frac{1}{2}$:

$$\sin\pi x = \pi x\prod_{i=1}^{\infty}(1 - \frac{x^2}{i^2})$$

Detailed proof using this outline can be found in [Kn].

Letting $n \to \infty$ in the formula above and using continuity of logarithmic function, we obtain

$$\lim_{n\to\infty}\log\frac{2\cdot 4\cdots 2n}{1\cdot 3\cdots(2n-1)}\frac{1}{\sqrt{2n+1}} = -\lim_{n\to\infty}\log(1+\frac{1}{2n})^{2n+1} + 1 - \log 2 + \lim_{n\to\infty}(2R(n) - R(2n+1))$$

Rewrite this as follows:

$$\log\frac{1}{\sqrt{2}}\lim_{n\to\infty}\frac{2\cdot 4\cdots 2n}{1\cdot 3\cdots(2n+1)}\frac{1}{\sqrt{n}} = C - \log 2$$

Thus, we have $C = \log\sqrt{2\pi}$. Substituting back to formula (14), we conclude with the Stirling's approximation formula

Theorem 3 (Stirling)**.**

$$n! = C(n)\sqrt{n}(\frac{n}{e})^n$$

where $\lim_{n\to\infty} C(n) = \sqrt{2\pi}$.

By the same approach as in the previous section, we would like to study a somewhat more general function $f(x) = \log(x+y)$ for $y > 0$. Set $a = 0$, $b = n$, where n is a positive integer, in the Euler-Maclaurin summation formula (10),

$$\sum_{i=0}^{n} \log(i+y) = ((n+y)\log(n+y) - n - y\log y) + \frac{1}{2}(\log(n+y) + \log y)) + \int_0^n \frac{P_1(t)}{t+y} dt$$

We showed earlier that

$$\log n! + \log n^y = (n + \frac{1}{2}) \log n - n + \log n^y + R(n)$$

(we simply added $\log n^y$ to the both sides of the formula (14).) Taking the difference of these two expressions, we obtain

$$\log \frac{n! n^y}{y(y+1)\cdots(y+n)} = (y - \frac{1}{2}) \log y - (n + y + \frac{1}{2}) \log \frac{n+y}{n} + R(n) - \int_0^n \frac{P_1(t)}{t+y} dt$$

Note that the limit of the left hand side as $n \to \infty$ is logarithm of the gamma function $\Gamma(y)$. Therefore, we discovered the following interesting relation:

Theorem 4.

$$\log \Gamma(y) = (y - \frac{1}{2}) \log y - y + \log \sqrt{2\pi} - \int_0^\infty \frac{P_1(t)}{t+y} dt$$

Proof. This follows almost immediately. It suffices to say that the expression $(n+y+\frac{1}{2}) \log \frac{n+y}{n} = (y+\frac{1}{2}) \log \frac{n+y}{n} + y \log(1+\frac{y}{n})^{\frac{n}{y}}$ tends to y whenever $n \to \infty$, and $\lim_{n\to\infty} R(n) = \log \sqrt{2\pi}$. $\square$

5 Generalizations of the Euler-Maclaurin Formula

We have two opposite ways to interpret the Euler-Maclaurin formula (10). In the way it is stated, it gives an estimation of the sum of values of the function f in terms of its integral. On the other hand, one can expresses the integral in terms of the sum as follows:

$$\begin{aligned} \int_a^b f(t)dt &= \sum_{n=a}^{b} f(n) - \frac{1}{2}(f(b) + f(a)) - \sum_{i=1}^{k} \frac{b_{2i}}{(2i)!} (f^{(2i-1)}(b) - f^{(2i-1)}(a)) - \\ &\quad - \int_a^b P_{2k+1}(t) f^{(2k+1)}(t) dt \end{aligned} \tag{17}$$

By the Mean value theorem, $\int_a^b P_{2k+1}(t) f^{(2k+1)}(t)dt = (b-a) P_{2k+1}(\alpha) f^{(2k+1)}(\alpha)$ for some $\alpha \in [a,b]$. Hence,

$$\begin{aligned} \int_a^b f(t)dt &= \sum_{n=a}^{b-1} \frac{1}{2}(f(n) + f(n+1)) - \sum_{i=1}^{k} \frac{b_{2i}}{(2i)!} (f^{(2i-1)}(b) - f^{(2i-1)}(a)) - \\ &\quad -(b-a) P_{2k+1}(\alpha) f^{(2k+1)}(\alpha) \end{aligned} \tag{18}$$

This formula formula exhibits the connection of the Euler-Maclaurin formula with the well-known trapezoid rule, which states that

$$\int_a^{a+h} f(t)dt \approx \frac{h}{2}(f(a) + f(a+h))$$

From the geometric point of view, it shows that the area underneath the graph of a function on the interval $[a, a+h]$ is approximately the area of the trapezoid with sides $(a, f(a))$ and $(a+h, f(a+h))$.

In the article [Sa], several interesting generalized formulas extending the formula (18) are justified. As an example, we state on these identities:

Theorem 5 (Sarafyan, Derr, Outlaw). *Assume* $\frac{b-a}{2} \in \mathbb{Z}_{>0}$ *and* $k \in \mathbb{Z}_{>1}$, *then there exists* $\alpha \in (a, b)$ *such that*

$$\begin{aligned} \int_a^b f(t)dt &= \frac{1}{3}(f(a) + 4f(a+1) + 2f(a+2) + \ldots + 2f(b-2) + 4f(b-1) + f(b)) + \\ &+ \sum_{i=1}^{k-1} \frac{(2^{2i}-4)b_{2i}}{3(2i)!}(f^{(2i-1)}(b) - f^{(2i-1)}(a)) + \\ &+ \frac{(b-a)(2^{2k}-4)b_{2k}}{3(2k)!} f^{2k}(\alpha) \end{aligned}$$

We shall not discuss the proof of this theorem in the paper. The basic idea is to use the Euler-Maclaurin identity twice with different step size, and then estimate the new remainder term using Mean value theorem. For details, we refer the reader to [Sa].

6 Conclusion

Euler-Maclaurin provides an important tool in estimating sums or integrals of functions. By means of it, one can derive many important identities in analysis relating important functions such as Riemann zeta function, gamma function, or Euler constant.

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