Week 5 (14 March, 2005): Circuit Components and Logical Connectives

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Examinator: dr. Hanno Wupper
Docenten: drs. Ling Cheung, drs. Martijn Hendriks, dr. Stijn Hoppenbrouwers

1 Notation

Throughout this exercise, we will use the symbol $\top$ (pronounced "top") to denote *true* and the symbol $\bot$ (pronounced "bottom") to denote *false*.

2 From Truth Functions to Circuit Components

So far in the course we have seen quite a number of connectives, e.g. $\neg$, $\wedge$ and $\vee$. Each connective can be seen as a function that maps truth values to truth values. For example, the connective $\wedge$ is a binary function f given by: $f(\bot,\bot) = \bot,\ f(\top,\bot) = \bot,\ f(\bot,\top) = \bot,$ and $f(\top,\top) = \top$.

Now let's think of f as a circuit component with two input wires and one output wire. We can interpret $\bot$ as a *low* signal and $\top$ as a *high* signal. Then f corresponds to the familiar **AND**-gate, whose output signal is *high* if and only if both input signals are *high*. Common circuit components/gates include: **AND, OR, inverter** (NOT), **XOR** (exclusive-OR), **NAND** (NOT-AND) and **NOR** (NOT-OR).

(If you didn't follow Machine Architecture last term, do a Google image search to see how these things are drawn.)

3 An Easy Problem

Construct an **AND**-gate using only **OR-** and **inverter**-gates. Similarly, construct an **OR**-gate using only **AND-** and **inverter**-gates.
Hint: think DeMorgan's Laws.

4 The Sheffer Stroke

The **NAND**-gate corresponds to a connective called the *Sheffer stroke*: $p \mid q := \neg(p \wedge q)$. For the following, assume you can duplicate signals by tabbing into a wire.

1. Construct an **inverter**-gate using only **NAND**-gates.
2. Construct an **OR**-gate using only **NAND**-gates.
3. Construct an **AND**-gate using only **NAND**-gates.

5 Solving Circuit Equations

Suppose you are given two **NAND**-gates. Construct a circuit with *no* inputs so that one of the wires in the circuit always has a *high* signal. Again you may duplicate signals.
Hint: you may feed the output of a gate back to the same gate.

(More on the reverse side.)

6 A Tedious Problem

Prove that you can construct any circuit with two input wires and one output wire using only **NAND**-gates.
Hint: there are $2^{2 \cdot 2} = 16$ such circuits.

7 Completeness of $\{|\}$

A set S of connectives is said to be *complete* if for any natural number n, any truth function of n arguments can be expressed using only connectives from S. Prove that the singleton set $\{|\}$ is complete.
Caution: this is a hard problem if you don't know mathematical induction.
Hint: use mathematical induction. For the base case, prove that we can obtain constant functions $\underline{\top}$ and $\underline{\bot}$ (cf. Problem 5). For the induction step, use Problem 6.